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ÖKLİD UZAYINDA ÖZEL EĞRİLERİN KARAKTERİZASYONLARI

OSMAN ÇAKIR

DOKTORA TEZİ

MATEMATİK ANABİLİM DALI

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TEZ BİLDİRİMİ

Tez yazım kurallarına uygun olarak hazırlanan ve kullanılan intihal tespit programının sonuçlarına göre; bu tezin yazılmasında bilimsel ahlak kurallarına uyulduğunu, başkalarının eserlerinden yararlanılması durumunda bilimsel normlara uygun olarak atıfta bulunulduğunu, tezin içerdiği yenilik ve sonuçların başka bir yerden alınmadığını, kullanılan verilerde herhangi bir tahrifat yapılmadığını, tezin herhangi bir kısmının bu üniversite veya başka bir üniversitedeki başka bir tez çalışması olarak sunulmadığını beyan ederim.

OSMAN ÇAKIR

Not: Bu tezde kullanılan özgün ve başka kaynaktan yapılan bildirişlerin, çizelge, şekil ve fotoğrafların kaynak gösterilmeden kullanımı, 5846 sayılı Fikir ve Sanat Eserleri Kanunundaki hükümlere tabidir.

ÖZET

ÖKLİD UZAYINDA ÖZEL EĞRİLERİN KARAKTERİZASYONLARI

OSMAN ÇAKIR

ORDU ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ

MATEMATİK ANABİLİM DALI

DOKTORA TEZİ, 161 SAYFA

(TEZ DANIŞMANI: DR. ÖĞR. ÜYESİ SÜLEYMAN ŞENYURT)

Bu tez dört bölüm halinde düzenlenmiştir. Giriş bölümünde çalışmanın amacı ve konunun ele alınma nedeni ifade edildi. Önceki Çalışmalar bölümünde, Öklid uzayında biharmonik eğrilerin sınıflandırılması ve diferensiyel denklemlerinin yazılması ile ilgili çalışmalara yer verildi. Genel Bilgiler bölümünde, tezde kullanılacak bazı tanım ve teoremler açıklandı.

Araştırma Bulguları bölümü, çalışmanın özgün kısmını oluşturmaktadır. Burada Öklid uzayında keyfi parametreyle verilen bir eğrinin diferensiyel denklemleri eğrinin teğet vektörüne ve Darboux vektörüne göre yazıldı. Eğrinin harmoniklik koşulları ise sırasıyla Laplace operatörüne göre (biharmonik veya 1. tipten harmonik) ve normal Laplace operatörüne göre (zayıf biharmonik veya 1. tipten harmonik) verildi. Sonra bu eğrinin bir involütü tanımlanarak, involüt eğrisinin diferensiyel denklemleri ve harmoniklik şartları involüt eğrisinin parametresine göre yazıldı. Daha sonra evolüt involüt eğrilerinin Frenet çatıları arasındaki geçiş matrisleri kullanılarak, involüt eğrisine ait yazılan denklemler ve harmoniklik koşulları esas eğrinin Frenet aparatları türünden ifade edildi. Benzer şekilde esas eğrinin bir Bertrand eğrisi olması halinde, bu eğrinin bir Bertrand partneri tanımlandı ve partner eğrisinin karakterizasyonları, involüt eğrisinde olduğu gibi esas eğrinin Frenet aparatları türünden yazıldı. Son olarak yazılan teoremlere ait sonuçlar verildi ve sayısal bir örnekle bu bölümdeki iddialar desteklendi.

Anahtar Kelimeler: Bertrand, Biharmonik, Darboux, Diferensiyel Denklem, İnvolüt, Laplace, Levi-Civita Konneksiyonu, Ortalama Eğrilik.

ABSTRACT

CHARACTERIZATIONS OF SPECIAL CURVES IN EUCLIDEAN SPACE

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MATHEMATICS

PHD THESIS, 161 PAGES

(SUPERVISOR: DR. ÖĞR. ÜYESİ SÜLEYMAN ŞENYURT)

This thesis consists of four fundamental chapters. In the Opening chapter, the purpose of the study and the reason for addressing the subject are revealed. Previous Studies section makes up of studies on classifications of biharmonic curves in Euclidean space and writing of differential equations. In the General Informations chapter, some definitions and theorems to be used in the research findings section are discussed.

Research Findings chapter constitutes the unique part of the present study. In this part, differential equations of a Frenet curve with arbitrary parameter are written first with respect to tangent vector and the Darboux vector of the given curve. Then the harmonicity conditions of the curve, that is, biharmonic or 1-type of harmonic with regard to Laplace operator and weak biharmonic or 1-type of harmonic according to normal Laplace operator are computed. Next, applying the properties of connected curves we define an involute of the given curve. By using the parameters of involute curve we first give all characterizations of involute curve itself. It follows that the differential equations and harmonicity conditions of the involute curve are expressed in terms of the Frenet apparatus of the main curve. Naturally we drive new conclusions. By the same manner, we suppose the main curve as a Bertrand curve and after writing the Bertrand partner curve, we give all characterizations of the Bertrand partner curve as in the case of involute curve. We complete the study with developing new corollaries and designating an example to strengthen our assertions.

Keywords: Bertrand, Biharmonic, Darboux, Involute, Differential Equation, Laplace, Levi-Civita Connection, Mean Curvature.

TEŞEKKÜR

Bu doktora çalışmasında, tez konumun belirlenmesinden yazılmasına kadar geçen tüm süreçlerde bilgi ve tecrübeleriyle, zaman ve mesai kavramı olmaksızın bana rehberlik yapan danışman hocam Sayın Dr. Öğr. Üyesi Süleyman ŞENYURT'a en samimi duygularımla teşekkür ederim.

Çalışmalarım boyunca maddi ve manevi desteklerini esirgemeyen Matematik Bölümü Başkanına ve bölümün kıymetli hocalarına, ayrıca canım aileme teşekkürü bir borç bilirim.

İÇİNDEKİLER

ŞEKİLLER LİSTESİ

Sayfa

SİMGELER ve KISALTMALAR LİSTESİ

$\mathbf{D_B}$	:	B Yönünde Kovaryant Türev
C	:	Birim Darboux Vektörü
$\boldsymbol{\tau}$	:	Burulma
$\mathbf{W}$	:	Darboux Vektörü
$\mathbf{C}$	:	Diferensiyellenebilir Fonksiyonların Cümlesi
$\boldsymbol{\kappa}$	:	Eğrilik
$\blacktriangledown$	:	Konneksiyon
$\boldsymbol{\Delta}$	:	Laplace Operatörü
$\mathbf{D}$	:	Levi-Civita Konneksiyonu
$\mathbf{D_N}$	:	N Yönünde Kovaryant Türev
$\Vert\ldots\Vert$	:	Norm
$\mathbf{D^{\perp}_B}$	:	Normal Konneksiyona Göre B Yönünde Türev
$\mathbf{D^{\perp}_N}$	:	Normal Konneksiyona Göre N Yönünde Türev
$\mathbf{D^{\perp}_T}$	:	Normal Konneksiyona Göre T Yönünde Türev
$\boldsymbol{\Delta}^{\perp}$	:	Laplace Operatörü
$\mathbf{H}$	:	Ortalama Eğrilik Vektörü
$\mathbf{E}$	:	Öklid Uzayı
$\mathbb{R}$	:	Reel Sayılar Uzayı
$\boldsymbol{R}$	:	Riemannian Manifoldu
$\mathbf{D_T}$	:	T Yönünde Kovaryant Türev
$\boldsymbol{\Sigma}$	:	Toplam Sembolü

1. GİRİŞ

Matematikte yorumladığımız pek çok şey, matematiğin bir alt dalı olan geometriyle ve geometrik olarak izah ettiğimiz kavramların büyük bir kısmı da aslında eğrilerle açıklanır. Bir yerin dünya üzerindeki coğrafi konumu enlemler ve boylamlarla belirlenir. Bu durum geometrik açıdan, Şekil 1.1 deki gibi küre üzerine çizilmiş eğrilerdir.

Şekil 1.1: Türkiye'nin Coğrafi Konumu

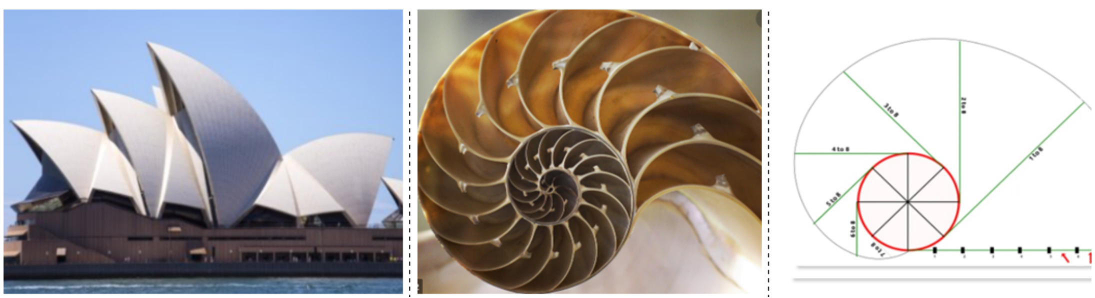
Otoyollarda ve kavşaklarda güvenli bir seyahat için hız limitinin belli olması esastır. Kavşaklardaki hız sınırı kavşağın eğrilik yarıçapını dikkate almakla mümkündür. Ayrıca Şekil 1.2 deki dairesel kavşakla otoyolun birleşme noktası, bir eğri ve bu eğrinin bir noktasındaki teğet vektörü olarak modellenebilir.

Şekil 1.2: Kuzey Marmara Otoyolundan Bir Kesit

Bölünmüş otoyolların şeritleri arasındaki mesafe hep sabittir. Şeritler arasındaki mesafenin sabit olması paralel eğrilerle, paralel eğriler ise Bertrand eğri çiftiyle ilgili bir durumdur. İki eğrinin karşılıklı noktalarında asli normalleri lineer bağımlı ise bu eğriler Bertrand eğri çifti oluşturur. Bertrand eğrileriyle ilgili daha fazla bilgi için (Matsuda ve Yorozu, 2003;

Babaarslan ve Yaylı, 2013; Camcı ve ark., 2020; Kuhnel, 2006; Öztekin ve Bektaş, 2010; Şenyurt ve As, 2013; Şenyurt ve Özgüner, 2013; Şenyurt ve Kılıçoğlu, 2017) kaynaklarına bakılabilir.

Doğada var olan ve sanat eserlerinde de pek çok uygulamalarını gördüğümüz altın oran geometrik açıdan bakıldığında involüt eğrilerini akla getirmektedir. Sanat eseri örneklerinden sadece birisi olan Sydney opera evinin inşaasında altın oran dikkate alınmıştır. Deniz helezonunda keşfedilen altın oranla involüt eğri modeli arasında da Şekil 1.3 te görüldüğü gibi yakın bir ilişki vardır. İki eğrinin karşılıklı noktalarında teğet vektörleri birbirine dik ise literatürde bu eğrilerden biri involüt diğeri de evolüt diye bilinir. Bu tür eğriler için (As ve Sarıoğlugil, 2014; Bilici ve Çalışkan, 1999; Bükcü ve Karacan, 2009; Şenyurt ve ark., 2016; Şenyurt ve Çakır, 2019) kaynaklarına bakılabilir.



Şekil 1.3: i)Sydney Opera Evi ii)Deniz Helezonu iii)İnvolüt Eğrisi

İnvolüt-evolüt eğrilerinin Frenet çatıları arasında geçiş matrişleri ile açıklanan bir bağıntı bulunmaktadır. Benzer şekilde Bertrand eğri çiftlerinin Frenet çatıları arasında da böyle bir ilişki vardır. Eğrilerde bu durum daha geniş bir açıdan ele alındığı taktirde ortalama eğrilik vektörü yardımıyla biharmonik eğrilerin sınıflandırılması yapılabilmektedir. Chen (1991), *Biharmonic Surface in Pseudo-Euclidean Spaces* başlıklı çalışmasında, *Öklid uzaylarındaki tek biharmonik altmanifoldlar minimal altmanifoldlardır,* önermesini verdi. Dahası $\mathbb{E}^n$ Öklid uzayında bir Riemann manifoldu M ve birim hızlı bir eğri $\gamma : I \longrightarrow M$ olmak üzere, γ eğrisinin H ortalama eğrilik vektörü

$$\Delta^k H + c_1 \Delta^{k-1} H + ... + c_{k-1} \Delta H + c_k H = 0 \tag{1.0.1}$$

diferensiyel denklemini sağlar. Burada Δ Laplace operatörü $c_1, c_2, \ldots, c_k$

$$
\begin{aligned}
c_1 &= -\sum_{t=p}^{q} \lambda_t\,, \\
c_2 &= \sum_{t<s} \lambda_t \lambda_s\,, \\
&. \\
&. \\
&. \\
c_{q-p+1} &= (-1)^{q-p+1} \sum_{t<k} \lambda_p ... \lambda_q\,, \quad (k = q-p+1) \ \text{ ve } \ \Delta x_i = \lambda_i x_i
\end{aligned}
$$

şeklinde birer katsayıdır. Daha sonra Dimitric (1992), bu iddianın doğruluğunu destekleyen bir çalışma yaptı.

Caddeo ve ark., (2005) biharmonik eğriler için geodeziklerin bir genelleştirmesidir ifadesini kullandı. Biharmonik eğriler, Riemannian manifoldunda dördüncü dereceden diferensiyel denklemlerin çözümüne denktir ve bu denklem

$$\nabla^3_{\gamma'}\gamma' - R(\gamma', \nabla_{\gamma'}\gamma')\gamma' = 0$$

şeklinde verildi. Burada R Riemannian manifoldunu ve ∇ konneksiyonu göstermektedir. Daha fazla bilgi için (Cho ve ark., 2007; Ferrandez ve ark., 1998; Kocayiğit ve Hacısalihoğlu, 2012; Külahcı, 2016; Nadjafikhah ve Mahdipour, 2011) kaynaklarına bakılabilir.

Bu tezde ilk olarak $\mathbf{E}^3$ üç boyutlu Öklid uzayında, keyfi parametreyle verilen bir eğrinin diferensiyel denklemleri Levi – Civita ve normal Levi – Civita konneksiyonuna göre yazıldı. Ardından verilen eğrinin harmoniklik koşulları, Laplace operatörüne göre (biharmonik veya 1.tipten harmonik) ve normal Laplace operatörüne göre (zayıf biharmonik veya 1.tipten harmonik) ayrı ayrı incelendi. Sonra bu eğrinin bir involütü tanımlanarak, involüt eğrisinin diferensiyel denklemleri ve harmoniklik koşulları involüt eğrisinin kendi parametresine göre yazıldı. Daha sonra evolüt-involüt eğrilerinin Frenet çatıları arasındaki geçişler kullanılarak, involüt eğrisine ait yazılan diferensiyel denklemler ve harmoniklik koşulları esas eğrinin Frenet aparatları türünden ifade edildi. Benzer şekilde esas eğrinin bir Bertrand eğrisi olması halinde, bu eğrinin Bertrand partner eğrisi yazılarak, partner eğrisinin diferensiyel denklemleri ve harmoniklik koşulları verildi. Yine bağlantılı eğrilerin Frenet çatıları arasındaki geçişler yardımıyla Bertrand partner eğrisine ait denklemler ve harmoniklik şartları esas eğrinin Frenet aparatları türünden ifade edildi. Son olarak teoremlere ait sonuçlar verildi ve sayısal bir örnekle bu bölümdeki iddialar desteklendi.

2. ÖNCEKİ ÇALIŞMALAR

Kocayiğit ve Hacısalihoğlu (2011), *"1-type curves and biharmonic curves in Euclidean 3-space"*, isimli çalışmasında, birim hızlı bir Frenet eğrisinin diferensiyel denklemini eğrinin birim teğet vektörüne göre

$$D_T^3 T - \Big(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\Big) D_T^2 T + \Big(\frac{-\kappa''}{\kappa} + \frac{\kappa'\tau'}{\kappa\tau} + 2(\frac{\kappa'}{\kappa})^2 + \kappa^2 + \tau^2\Big) D_T T + \kappa\tau\big(\frac{\kappa}{\tau}\big)' T = 0 \tag{2.0.1}$$

şeklinde ifade etti. H ortalama eğriliği ve Δ Laplace operatörünü göstermek üzere, eğrinin biharmonik olma şartını $\Delta H = 0$ ve 1.tipten harmonik olma şartını da $\Delta H = \lambda H$, $\lambda \neq 0$, şeklinde verdi. Buna göre birim hızlı bir Frenet eğrisinin harmoniklik koşulları:

i) Birim hızlı bir Frenet eğrisinin biharmonik olması için gerek ve yeter koşul

$$\kappa\kappa' = 0, \quad \kappa\tau^2 + \kappa^3 - \kappa'' = 0, \quad 2\kappa'\tau + \kappa\tau' = 0,$$

ii) Birim hızlı bir Frenet eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$\kappa\kappa' = 0, \quad \kappa\tau^2 + \kappa^3 - \kappa'' = \lambda\kappa, \quad 2\kappa'\tau + \kappa\tau' = 0$$

olmasıdır.

Kocayiğit ve ark., (2014) *"Harmonic 1-type curves and weak biharmonic curves in Lorentzian 3-space"* isimli çalışmasında, birim hızlı bir Frenet eğrisini karakterize eden diferensiyel denklemleri normal konneksiyona göre

$$
\begin{aligned}
&1) \quad \tau\big(\nabla^{\perp}_{\gamma'}\big)^2 N - \tau'\nabla^{\perp}_{\gamma'} N - \varepsilon_1\tau^3 N = 0,\\[2ex]
&2) \quad \tau\big(\nabla^{\perp}_{\gamma'}\big)^2 B - \tau'\nabla^{\perp}_{\gamma'} B - \varepsilon_1\tau^3 B = 0,\\[2ex]
&3) \quad \Delta^{\perp}_{\gamma'} H + \big(\frac{3\kappa'}{\kappa}\big)\nabla^{\perp}_{\gamma'} H - \Big(3\big(\frac{\kappa'}{\kappa}\big)^2 - \frac{\kappa^2\tau + \kappa''}{\kappa}\Big) H = 0,
\end{aligned}
\tag{2.0.2}$$

şeklinde ve bu eğrinin harmoniklik koşullarını

i) Birim hızlı bir Frenet eğrisinin zayıf biharmonik olması için gerek ve yeter koşul

$$\varepsilon_1\varepsilon_2\kappa\tau^2 - \varepsilon_1\varepsilon_3\kappa'' = 0, \quad 2\kappa'\tau + \kappa\tau' = 0,$$

ii) Birim hızlı bir Frenet eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$\varepsilon_1\varepsilon_2\kappa\tau^2 - \varepsilon_1\varepsilon_3\kappa'' = \varepsilon_3\lambda\kappa, \quad 2\kappa'\tau + \kappa\tau' = 0$$

şeklinde verdi. Burada $\nabla^\perp$ normal konneksiyona göre türev operatörünü ve ε_i vektörün özdeğerini göstermektedir.

Arslan ve ark., (2016) *"Characterizations of space curves with 1-type Darboux instantaneous rotation vector"*, isimli çalışmasında biharmonik eğrilerin diferensiyel denklemini konneksiyona ve normal konneksiyona göre sırasıyla,

$$\begin{aligned}&\Big(\kappa\tau' - \kappa'\tau\Big)^2 D_T^3 W + \Big(2\big(\kappa\tau' - \kappa'\tau\big)'\big(\kappa'\tau - \kappa\tau'\big)\Big) D_T^2 W + \Big(2\big((\kappa\tau' - \kappa'\tau)'\big)^2\\
&+\tau\big(\kappa\tau' - \kappa'\tau\big)\big(\tau(\kappa\tau' - \kappa'\tau) + \kappa''\big) - \kappa\big(\kappa\tau' - \kappa'\tau\big)\big(\tau''' - \kappa(\kappa\tau' - \kappa'\tau)\big)\Big) D_T W\\
&+\Big(\kappa'\big(\kappa\tau' - \kappa'\tau\big)\big(\tau''' - \kappa(\kappa\tau' - \kappa'\tau)\big) - \tau'\big(\kappa\tau' - \kappa'\tau\big)\big(\tau(\kappa\tau' - \kappa'\tau) + \kappa''\big)\\
&-2\big(\kappa\tau' - \kappa'\tau\big)'\big(\kappa'\tau'' - \kappa''\tau'\big)\Big) W = 0,\end{aligned} \tag{2.0.3}$$

$$\kappa^2\tau\big(\nabla^\perp_{\gamma'}\big)^2 W^\perp - \big(\kappa(\kappa'\tau + (\kappa\tau)')\big)\nabla^\perp_{\gamma'} W^\perp + \Big(\kappa'\big(\kappa'\tau + (\kappa\tau)'\big) + \kappa\tau\big(\kappa\tau^2 - \kappa''\big)\Big) W^\perp = 0 \tag{2.0.4}$$

şeklinde verdi. Aynı çalışmada birim hızlı Frenet eğrisinin konneksiyona göre 1.tipten harmonik olma şartını,

$$\tau'' = -\lambda\tau, \quad \kappa\tau' - \kappa'\tau = 0, \quad \kappa'' = -\lambda\kappa$$

ve normal konneksiyona göre 1.tipten harmonik olma koşulunu ise

$$\kappa'' + (\lambda - \tau^2)\kappa = 0, \quad 2\kappa'\tau + \tau'\kappa = 0$$

şeklinde verdi.

Şenyurt ve Çakır (2018), " *Differential Equations for a Space Curve According to the Unit Darboux Vector* " isimli çalışmasında birim hızlı bir Frenet eğrisinin diferensiyel denklemlerini konneksiyona ve normal konneksiyona göre sırasıyla,

$$\nabla^3_{\alpha'}C - \Big(\frac{\varphi''}{\varphi'} + \frac{(\varphi' \parallel W \parallel)'}{\parallel W \parallel}\Big)\nabla^2_{\alpha'}C - \Big((\frac{\varphi''}{\varphi'})' - ((\varphi')^2 + \parallel W \parallel^2) - \frac{\varphi''}{\varphi'}\frac{(\varphi' \parallel W \parallel)'}{\parallel W \parallel}\Big)\nabla_{\alpha'}C$$

$$-\Big(\frac{\varphi'}{\parallel W \parallel}(\varphi' \parallel W \parallel)' - ((\varphi')^2)'\Big)C = 0 \tag{2.0.5}$$

ve

$$\Big(\tau cos\varphi\Big)\nabla^{\perp(2)}_{\alpha'}C^{\perp} - \Big(\varphi'\tau sin\varphi - (\tau cos\varphi)'\Big)(\nabla^{\perp}_{\alpha'})C^{\perp}$$

$$+\Big(\big(\varphi'\tau sin\varphi - (\tau cos\varphi)'\big)\varphi' tan\varphi - \tau^3 cos\varphi - \tau(\varphi' sin\varphi)'\Big)C^{\perp} = 0 \tag{2.0.6}$$

şeklinde verdi. Burada φ eğrinin Darboux vektörü ile binormali arasındaki açıdır. Birim Darboux vektörüne göre eğrinin harmoniklik koşullarını

i) Birim hızlı bir Frenet eğrisinin biharmonik olması için gerek ve yeter koşul $\varphi' = 0$,

ii) Birim hızlı bir Frenet eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$\lambda = \tau^2 + \Big(\frac{\tau'}{2\tau}cos\varphi\Big)' sec\varphi$$

şeklinde verdi.

3. GENEL BİLGİLER

3.1 Öklid Uzayı ve Eğriler

$\mathbb{R}^3$, 3-boyutlu standart reel vektör uzayı olsun. $\forall\ X,\ Y\ \in \mathbb{R}^3$ için,

$$\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \quad \langle X, Y \rangle = \sum_{i=1}^{3} x_i y_i$$

şeklinde tanımlı fonksiyon bir iç çarpım fonksiyonudur. $(\mathbb{R}^3, \langle , \rangle)$ ikilisine bir **Öklid uzayı** denir ve $\mathbf{E}^3$ ile gösterilir. $\mathbf{E}^3$ ün bir P noktasındaki tanjant vektörlerinin cümlesi $T_{E^3}(P) = \{\overrightarrow{v_p} \mid p \in \mathbf{E}^3, \overrightarrow{v} \in \mathbb{R}^3\}$ ile gösterilir ve bu uzaya reel sayılar cismi üzerinde **tanjant uzayı** denir. $\mathbf{E}^3$ ün bütün noktalarındaki tanjant uzaylarının cümlesi $\mathbf{E}^3$ ile gösterilir ve bu uzaya **vektör alanları uzayı** denir (Hacısalihoğlu, 1988).

Tanım 3.1.1 $f : \mathbf{E}^3 \to \mathbb{R}$ diferensiyellenebilir dönüşümü ve $\overrightarrow{v}_p \in T_{\mathbf{E}^3}(P)$ tanjant vektörü verilsin. P, $Q \in \mathbf{E}^3$ ve $\overrightarrow{v}_p = \overrightarrow{PQ}$ için,

$$\overrightarrow{v}_p(f) = \frac{d}{dt}\big[f(p_1 + t(q_1 - p_1)),\ \ f(p_2 + t(q_2 - p_2)),\ \ f(p_3 + t(q_3 - p_3))\big]\ |_{t=0}$$

türevine f fonksiyonunun $\overrightarrow{v}_p$ **tanjant vektörü yönündeki türevi** denir. $X \in \mathbf{E}^3$, $f \in \mathbf{C}(\mathbf{E}^3, \mathbb{R})$ ve $(X(f))(P) = X_p(f)$ olmak üzere, $X(f) \in \mathbf{C}(\mathbf{E}^3, \mathbb{R})$ fonksiyonuna f nin X yönündeki **yöne göre türevi** denir.
Bu tanıma göre $\forall\, X, Y \in \chi(\mathbf{E}^3)$, $\forall\, f,\ g,\ h \in \mathbf{C}(\mathbf{E}^3, \mathbb{R})$, $\forall\, a, b \in \mathbb{R}$, $P \in \mathbf{E}^3$ ve $(fX)(P) = f(P)X_p$ olmak üzere, aşağıdaki önermeler doğrudur.

1. $(fX + gY)(h) = fX(h) + gY(h)$,

2. $X(af + bg) = aX(f) + bX(g)$,

3. $X(fg) = X(f)g + fX(g)$, (Hacısalihoğlu, 1988).

Tanım 3.1.2 α diferensiyellenebilir bir eğri olsun.

$$D : T_{\mathbf{E}^3}(\alpha(t)) \times \mathbf{C}(\mathbf{E}^3, \mathbb{R}) \to \mathbb{R} \ , \qquad D_{\alpha'(t)} f = \alpha'(t)(f)$$

şeklinde tanımlanan D operatörüne **kovaryant türev operatörü** ve $\alpha'(t)(f) \in \mathbb{R}$ değerine de f fonksiyonunun α eğrisi boyunca **kovaryant türevi** denir. $Y = (y_1, y_2, y_3)$, $y_i : \mathbf{E}^3 \longrightarrow \mathbb{R}$, $1 \leq i \leq 3$, koordinat fonksiyonları $\mathbf{C}^\infty$ sınıfından diferensiyellenebilir olmak üzere, Y nin X vektör alanına göre kovaryant türevi

$$D_X Y = (X_p[y_1],\ X_p[y_2],\ X_p[y_3]).$$

X, Y, Z ve W vektörleri $\mathbf{C}^\infty$ sınıfından olmak üzere aşağıdaki önermeler doğrudur.

$$\begin{aligned} D_X(Y+Z) &= D_X Y + D_X Z, \\ D_{X+W}(Y) &= D_X Y + D_W Y, \\ D_{f(P)X}(Y) &= f(P) D_X Y,\ f : \mathbf{E}^3 \to \mathbb{R},\ \ P \in \mathbf{E}^3, \\ D_X(fY) &= X(f)Y + f D_X Y,\ f \in \mathbf{C}(\mathbf{E}^3, \mathbb{R}),\ \text{(Hacısalihoğlu, 1988).} \end{aligned} \tag{3.1.1}$$

Tanım 3.1.3 α diferensiyellenebilir bir eğri olsun. Buna göre
a) α birim hızlı bir eğri ise Frenet vektörleri, eğrilikleri ve Frenet formülleri sırasıyla

$$T(s) = \alpha'(s), \quad N(s) = \frac{\alpha''(s)}{\| \alpha''(s) \|}, \quad B(s) = T(s) \wedge N(s),$$

$$\kappa(s) = \| T'(s) \|, \quad \tau(s) = -\langle B'(s),\ N(s) \rangle \quad \text{ve}$$

$$D_T T = \kappa N, \qquad D_T N = -\kappa T + \tau B, \qquad D_T B = -\tau N \tag{3.1.2}$$

b) α keyfi parametreli bir eğri ise Frenet vektörleri, eğrilikleri ve Frenet formülleri sırasıyla

$$T = \frac{\alpha'}{\| \alpha' \|}, \quad N = B \times T, \quad B = \frac{\alpha' \times \alpha''}{\| \alpha' \times \alpha'' \|},$$

$$\kappa = \frac{\| \alpha' \times \alpha'' \|}{\| \alpha' \|^3}, \quad \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\| \alpha' \times \alpha'' \|^2} \tag{3.1.3}$$

$$D_T T = \vartheta\kappa N, \quad D_T N = \vartheta(-\kappa T + \tau B), \quad D_T B = -\vartheta\tau N \tag{3.1.4}$$

bağıntısıyla verilir. Burada T ye **birim teğet**, N ye **asli normal** ve B ye de **binormal vektör** denir. Eğer $\kappa \neq 0$ ise α eğrisine **Frenet eğrisi** adı verilir (Hacısalihoğlu, 1988).

Tanım 3.1.4 Bir α eğrisinin T, N, B bir eksen etrafında, ani bir helis hareketi yaptığı kabul edilir. Bu eksene **Darboux ekseni** denir ve bu vektör

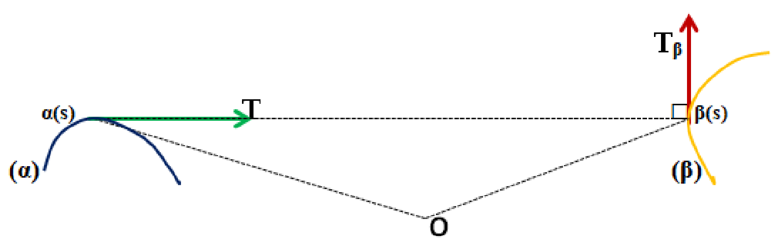
$$W = \tau T + \kappa B \tag{3.1.5}$$

bağıntısıyla verilir (Hacısalihoğlu, 1988).
W ile B vektör alanları arasındaki açı ϕ ile gösterilirse C birim Darboux vektörü

$$C = sin\phi T + cos\phi B\,, \quad sin\phi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}},\ cos\phi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} \tag{3.1.6}$$

bağıntısıyla verilir (Fenchel, 1951).

Tanım 3.1.5 α birim hızlı olmak üzere, diferensiyellenebilir iki eğri α ve β olsun. α eğrisinin $\alpha(s)$ noktasındaki teğeti β eğrisinin $\beta(s)$ noktasındaki teğetine dik ise β eğrisine α eğrisinin bir **involütü** denir.



Şekil 3.1: α-Evolüt Eğrisi ve β-İnvolüt Eğrisi

Bu tanıma göre involüt eğrisi

$$\beta(s) = \alpha(s) + \lambda(s)T(s)\,, \quad \lambda(s) = c - s\,, \quad c \in \mathbb{R}$$

bağıntısıyla verilir. β involüt eğrisinin Frenet vektörleri T_β, N_β, B_β ile gösterilirse, β eğrisiyle esas eğrinin Frenet vektörleri arasında

$$T_\beta = N, \quad N_\beta = \frac{1}{\sqrt{\kappa^2+\tau^2}}(-\kappa T + \tau B), \quad B_\beta = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\tau T + \kappa B) \tag{3.1.7}$$

bağıntısı vardır. β eğrisinin κ_β ve τ_β eğrilikleriyle, α eğrisinin κ ve τ eğrilikleri arasında

$$\kappa_\beta(s) = \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\ , \qquad \tau_\beta(s) = \frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)} \tag{3.1.8}$$

bağıntısı vardır (Sabuncuoğlu, 2014).

β involüt eğrisinin Darboux vektörü W_β ve birim Darboux vektörü C_β ile gösterilirse bu vektörler esas eğrinin Frenet aparatları türünden sırasıyla,

$$W_\beta = \frac{1}{\lambda\kappa}W + \frac{\phi'}{\lambda\kappa}N\,, \tag{3.1.9}$$

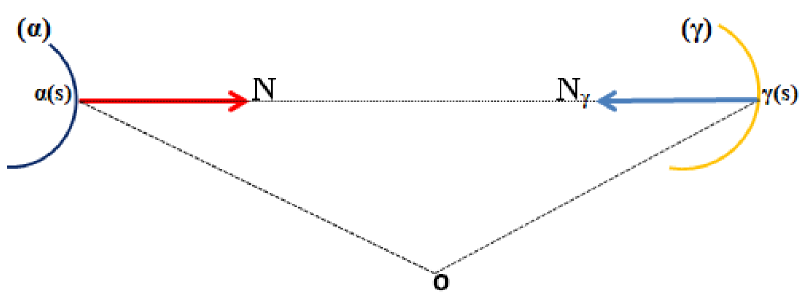
$$C_\beta = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}N + \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}C \tag{3.1.10}$$

bağıntılarıyla verilir (Şenyurt ve ark., 2016).

Tanım 3.1.6 α birim hızlı olmak üzere, diferensiyellenebilir iki eğri α ve γ olsun. α eğrisi ile γ eğrisinin asli normalleri lineer bağımlı ise α ya bir Bertrand eğrisi, γ ya ise Bertrand partner eğrisi, (α, γ) ikilisine de **Bertrand eğri çifti** denir. Buna göre Bertrand eğrisi

$$\gamma(s) = \alpha(s) + \lambda(s)N(s)$$

bağıntısıyla verilir. Bertrand eğri çiftlerinin karşılıklı noktaları arasındaki uzaklık sabittir.



Şekil 3.2: $(\alpha\,,\gamma)$ Bertrand Eğri Çifti

α eğrisinin birim teğet vektörü T ile γ partner eğrisinin birim teğet vektörü T_γ arasındaki açı da sabittir (Sabuncuoğlu, 2014).

(α, γ) Bertrand eğri çifti ve bu eğrilerin Frenet vektörleri, sırasıyla T, N, B ve $T_\gamma, N_\gamma, B_\gamma$ olsun. Bu vektörler arasında,

$$T_\gamma = cos\theta T + sin\theta B, \qquad N_\gamma = N, \qquad B_\gamma = -sin\theta T + cos\theta B \tag{3.1.11}$$

bağıntısı vardır. Burada θ Bertrand eğri çiftlerinin teğet vektörleri arasındaki açıdır. γ partner eğrisinin κ_γ ve τ_γ eğrilikleri ile α eğrisinin κ ve τ eğrilikleri arasında

$$\kappa_\gamma = \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}\ , \qquad \tau_\gamma = \frac{1}{\lambda^2\tau} sin^2\theta\,, \quad \lambda \in \mathbb{R} \tag{3.1.12}$$

bağıntısı vardır. $(\alpha,\ \gamma)$ ikilisinin bir Bertrand çifti olması için gerek ve yeter koşul

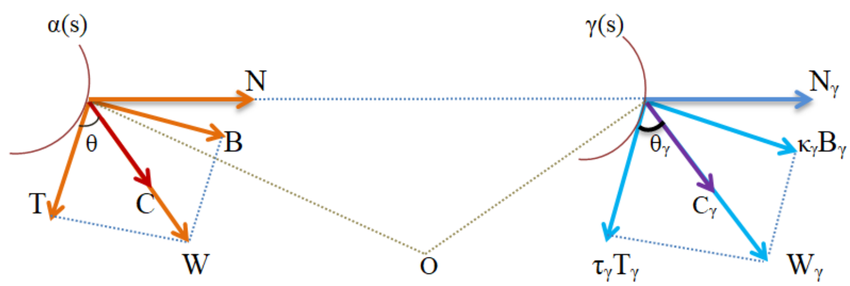
$$\lambda\kappa + \mu\tau = 1, \quad \mu = \lambda ctg\theta \ \text{ ve } \ \lambda,\ \mu \in \mathbb{R} \tag{3.1.13}$$

bağıntısıyla verilir (Sabuncuoğlu, 2014).

$(\alpha,\ \gamma)$ Bertrand eğri çiftinde partner eğrisinin W_γ Darboux vektörü ile Bertrand eğrisinin W Darboux vektörü arasındaki geçiş

$$W_\gamma = \frac{1}{\tau(\lambda^2 + \mu^2)} W \tag{3.1.14}$$

bağıntısı ile verilir (Çelik, 2016).



Şekil 3.3: $(\alpha\,,\gamma)$ Bertrand Eğri Çiftinin Darboux Vektörleri

$(\alpha,\ \gamma)$ Bertrand eğri çiftinde partner eğrisinin Şekil 3.3 te gösterilen θ_γ açısı, Bertrand eğrisinin Frenet aparatları cinsinden ifadesi

$$tan\theta_\gamma \ = \ (\frac{1}{\mu\kappa - \lambda\tau}) \ \text{ ve } \ \theta_\gamma = arctan(\frac{1}{\mu\kappa - \lambda\tau}) \tag{3.1.15}$$

şeklinde verilir (Çelik, 2016).

3.2 Eğrinin Harmonikliği ve Diferensiyel Denklemleri

Tanım 3.2.1 $\mathbf{E}^3$ ün Levi – Civita konneksiyonu D olmak üzere,

$$H = D_T T = \kappa N \tag{3.2.1}$$

ifadesine **ortalama eğrilik vektör alanı** denir (Chen, 1991).

Tanım 3.2.2 Laplace Operatörü: H, α eğrisi boyunca ortalama eğrilik vektör alanı ve $\Delta : \chi^{\perp}(\alpha(I)) \to \chi(\alpha(I))$ olsun. H nin Laplace dönüşümü

$$\Delta H = -D_T^2 H \tag{3.2.2}$$

şeklinde tanımlanır ve Δ ya da **Laplace operatörü** denir (Chen, 1991).

Teorem 3.2.1 Birim hızlı α eğrisinin ortalama eğrilik vektör alanı H ve Laplace dönüşümü ΔH olmak üzere, Kocayiğit ve Hacısalihoğlu, (2011)

1. $\Delta H = 0$ ise α eğrisi biharmoniktir.

2. $\Delta H = \lambda_0 H$ ise α eğrisi 1. tipten harmoniktir, $\lambda_0 \in \mathbb{R}$.

Tanım 3.2.3 Normal Laplace Operatörü: Birim hızlı α eğrisinin normal demeti $\chi^{\perp}(\alpha(I))$ olsun. $\forall X \in \chi^{\perp}(\alpha(I))$ için $D^{\perp}$ normal konneksiyonu ve $\Delta^{\perp}$ normal Laplace operatörünü göstermek üzere, **normal konneksiyon** ve **normal Laplace dönüşümü** sırasıyla,

$$D^{\perp} : \chi^{\perp}(\alpha(I)) \to \chi^{\perp}(\alpha(I)) \,, \quad D^{\perp} X = D_T X - \left\langle D_T X, T \right\rangle T \,, \tag{3.2.3}$$

$$\Delta^{\perp} : \chi^{\perp}(\alpha(I)) \to \chi^{\perp}(\alpha(I)) \,, \quad \Delta^{\perp} X = -D_T^{\perp} D_T^{\perp} X \tag{3.2.4}$$

şeklinde tanımlanır (Dimitric, 1992).

Teorem 3.2.2 Birim hızlı α eğrisinin ortalama eğrilik vektör alanı H ve bu vektörün normal Laplace dönüşümü $\Delta^{\perp} H$ olmak üzere, Hacısalihoğlu ve ark., (2014)

1. $\Delta^{\perp} H = 0$ ise α eğrisi zayıf biharmoniktir.

2. $\Delta^{\perp} H = \lambda_0 H$ ise α eğrisi 1. tipten harmoniktir, $\lambda_0 \in \mathbb{R}$.

Teorem 3.2.3 α birim hızlı bir Frenet eğrisi olsun. α eğrisinin T teğet vektörüne göre diferensiyel denklemi

$$D_T^3 T - \big(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\big) D_T^2 T + \big(-\frac{\kappa''}{\kappa} + \frac{\kappa'}{\kappa}\frac{\tau'}{\tau} + 2(\frac{\kappa'}{\kappa})^2 + \kappa^2 + \tau^2\big) D_T T + \kappa\tau\big(\frac{\kappa}{\tau}\big)' T = 0$$

bağıntısıyla verilir (Kocayiğit ve Hacısalihoğlu, 2011).

Teorem 3.2.4 α birim hızlı bir Frenet eğrisi olsun. α eğrisinin N asli normal vektörüne veya B binormal vektörüne göre diferensiyel denklemi

$$D_T^{\perp} D_T^{\perp} N - \big(\frac{\tau'}{\tau}\big) D_T^{\perp} N + \tau^2 N = 0$$

veya

$$D_T^{\perp} D_T^{\perp} B - \big(\frac{\tau'}{\tau}\big) D_T^{\perp} B + \tau^2 B = 0$$

bağıntısıyla verilir (Hacısalihoğlu ve ark., 2014).

Teorem 3.2.5 α birim hızlı bir Frenet eğrisi olsun. α eğrisini W Darboux vektörüne göre diferensiyel denklemi

$$\lambda_4 D_T^3 W + \lambda_3 D_T^2 W + \lambda_2 D_T W + \lambda_1 W = 0$$

bağıntısıyla verilir. Burada $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$

$$\begin{aligned}
\lambda_1 &= \kappa' f(\tau''' - \kappa f) - \tau' f(\tau f + \kappa'') - 2g(\kappa'\tau'' - \tau'\kappa''), \\
\\
\lambda_2 &= 2g^2 + \tau f(\tau f + \kappa'') - \kappa f(\tau''' - \kappa f), \\
\\
\lambda_3 &= -2fg, \\
\\
\lambda_4 &= f^2 \ \text{ ve } \ f = \kappa\tau' - \kappa'\tau, \quad g = f'
\end{aligned}$$

şeklinde birer katsayıdır (Arslan ve ark., 2016).

Teorem 3.2.6 α birim hızlı bir Frenet eğrisi olsun. α eğrisini normal konneksiyona göre karakterize eden diferensiyel denklem

$$k_3(D_T^\perp)^2 W^\perp + k_2 D_T^\perp W^\perp + k_1 W^\perp = 0$$

bağıntısıyla verilir. Burada $k_1,\ k_2,\ k_3$

$$\begin{aligned}
k_1 &= \kappa'\big(\kappa'\tau + (\kappa\tau)'\big) - \kappa\tau\big(\kappa'' - \kappa\tau^2\big), \\
k_2 &= -\kappa\big(\kappa'\tau + (\kappa\tau)'\big), \\
k_3 &= \kappa^2\tau
\end{aligned}$$

şeklinde birer katsayıdır (Arslan ve ark., 2016).

Teorem 3.2.7 Birim hızlı α eğrisinin Darboux vektörü W ve bu vektörün Laplace dönüşümü ΔW ile gösterilsin. Konneksiyona göre α eğrisinin harmoniklik koşulları, (Arslan ve ark., 2016)

1. $\Delta W = 0$ ise α eğrisi Darboux vektörüne göre biharmoniktir.

2. $\Delta W = \lambda_0 W$ ise α eğrisi Darboux vektörüne göre 1. tipten harmoniktir.

Teorem 3.2.8 Birim hızlı bir α eğrisinin Darboux vektörü W ve bu vektörün normal Laplace dönüşümü $\Delta^\perp W^\perp$ ile gösterilsin. Normal konneksiyona göre α eğrisinin zayıf biharmonik ve 1.tipten harmonik olma koşulları, (Arslan ve ark., 2016)

1. $\Delta^\perp W^\perp = 0$ ise α eğrisi Darboux vektörüne göre zayıf biharmoniktir.

2. $\Delta^\perp W^\perp = \lambda_0 W^\perp$ ise α eğrisi Darboux vektörüne göre 1. tipten harmoniktir.

4. ARAŞTIRMA BULGULARI

Üç boyutlu Öklid uzayında birim hızlı bir Frenet eğrisinin, Laplace operatörüne göre harmoniklik koşulları ve Levi-Civita konneksiyonuna göre diferensiyel denklemlerinin yazılışı bilinmektedir. Çalışmanın bu bölümünde, önce keyfi parametreyle verilen bir eğrinin diferensiyel denklemleri eğrinin teğet vektörüne ve Darboux vektörüne göre yazıldı. Eğrinin harmoniklik koşulları sırasıyla Laplace operatörüne göre (biharmonik veya 1. tipten harmonik) ve normal Laplace operatörüne göre (zayıf biharmonik veya 1. tipten harmonik) verildi. Sonra eğrinin bir involütü tanımlanarak involüt eğrisinin denklemleri ve harmoniklik şartları involüt eğrisinin parametresine göre yazıldı. Daha sonra evolüt ve involüt eğrilerinin Frenet çatıları arasındaki geçiş matrisleri kullanılarak, involüt eğrisinin diferensiyel denklemleri ve harmoniklik şartları esas eğrinin Frenet aparatları türünden ifade edildi. Benzer şekilde esas eğrinin bir Bertrand eğrisi olması halinde, bu eğrinin bir Bertrand partneri yazılarak, partner eğrisinin karakterizasyonları Bertrand eğrisinin Frenet aparatları türünden ifade edildi. Son olarak yazılan teoremlere ait sonuçlar verildi ve sayısal bir örnekle bu bölümdeki iddialar desteklendi.

Şimdi birim hızlı olmayan herhangi bir eğrinin harmoniklik koşullarını ve diferensiyel denklemlerinin yazılışını verelim.

Lemma 4.0.1 Birim hızlı olmayan bir α eğrisinin konneksiyona göre biharmonik olma ve 1.tipten harmonik olma koşulları aşağıdaki gibi verilir:

1. α eğrisi biharmoniktir ancak ve ancak
$3\big(\vartheta\kappa\big)'\vartheta\kappa = 0,\quad \big(\vartheta\kappa\big)^3 + \vartheta\kappa\big(\vartheta\tau\big)^2 - \big(\vartheta\kappa\big)'' = 0$ ve $-2\big(\vartheta\kappa\big)'\vartheta\tau - \vartheta\kappa\big(\vartheta\tau\big)' = 0.$

2. α eğrisi 1.tipten harmoniktir ancak ve ancak
$3\big(\vartheta\kappa\big)'\vartheta\kappa = 0,\quad \big(\vartheta\kappa\big)^3 + \vartheta\kappa\big(\vartheta\tau\big)^2 - \big(\vartheta\kappa\big)'' = \lambda_0\vartheta\kappa$ ve $-2\big(\vartheta\kappa\big)'\vartheta\tau - \vartheta\kappa\big(\vartheta\tau\big)' = 0\,,$
$\lambda_0 \in \mathbb{R},\ \ \vartheta = \parallel \frac{d}{ds}\alpha(s) \parallel.$

İspat. (3.2.1) bağıntısından $H = \vartheta\kappa N$ dir. (3.2.2) bağıntısına göre H ortalama eğrilik vektörünün Laplace görüntüsü

$$\begin{aligned}
\Delta H &= -D_T^2\big(\vartheta\kappa N\big) \\
&= -D_T\big(D_T(\vartheta\kappa N)\big) \\
&= -D_T\big((\vartheta\kappa)'N + (\vartheta\kappa)D_T N\big) \\
&= -\big((\vartheta\kappa)''N + 2(\vartheta\kappa)'D_T N + \vartheta\kappa D_T^2 N\big) \\
&= -\Big((\vartheta\kappa)''N + 2(\vartheta\kappa)'(-\vartheta\kappa T + \vartheta\tau B) + \vartheta\kappa(-\vartheta\kappa)'T \\
&\quad (-\vartheta\kappa)\vartheta\kappa N + (\vartheta\tau)'B + \vartheta\tau(-\vartheta\tau)N\Big) \\
&= \Big(3\big(\vartheta\kappa\big)'\vartheta\kappa\Big)T + \Big(\big(\vartheta\kappa\big)^3 + \vartheta\kappa\big(\vartheta\tau\big)^2 - \big(\vartheta\kappa\big)''\Big)N \\
&\quad \Big(-2\big(\vartheta\kappa\big)'\vartheta\tau - \vartheta\kappa\big(\vartheta\tau\big)'\Big)B
\end{aligned}$$

şeklinde bulunur. $\Delta H = 0$ ise

$$3\big(\vartheta\kappa\big)'\vartheta\kappa = 0, \quad \big(\vartheta\kappa\big)^3 + \vartheta\kappa\big(\vartheta\tau\big)^2 - \big(\vartheta\kappa\big)'' = 0 \text{ ve } 2\big(\vartheta\kappa\big)'\vartheta\tau + \vartheta\kappa\big(\vartheta\tau\big)' = 0$$

birinci önerme, $\Delta H = \lambda_0 H$ ise

$$3\big(\vartheta\kappa\big)'\vartheta\kappa = 0, \quad \big(\vartheta\kappa\big)^3 + \vartheta\kappa\big(\vartheta\tau\big)^2 - \big(\vartheta\kappa\big)'' = \lambda_0\vartheta\kappa \text{ ve } 2\big(\vartheta\kappa\big)'\vartheta\tau + \vartheta\kappa\big(\vartheta\tau\big)' = 0$$

ikinci önerme sağlanır. □

Lemma 4.0.2 Birim hızlı olmayan bir α eğrisinin normal konneksiyona göre zayıf biharmonik olma ve 1.tipten harmonik olma koşulları aşağıdaki gibi verilir:

1. α eğrisi zayıf biharmoniktir ancak ve ancak

$$\vartheta^3\big(\tau\big)^2\kappa - \big(\vartheta\kappa\big)'' = 0 \quad \text{ve} \quad \big(\vartheta\kappa\big)'\vartheta\tau + \big(\vartheta^2\kappa\tau\big)' = 0\,.$$

2. α eğrisi 1.tipten harmoniktir ancak ve ancak

$$\vartheta^3\big(\tau\big)^2\kappa - \big(\vartheta\kappa\big)'' = \lambda_0\kappa \quad \text{ve} \quad \big(\vartheta\kappa\big)'\vartheta\tau + \big(\vartheta^2\kappa\tau\big)' = 0\,, \quad \lambda_0 \in \mathbb{R}\,.$$

İspat. (3.2.4) bağıntısından H ortalama eğrilik vektörünün normal Laplace görüntüsünü hesaplamak için önce H nin birinci türevi alınırsa

$$\begin{aligned} D_T H &= D_T\big(\vartheta\kappa N\big) \\ &= \big(\vartheta\kappa\big)' N + \vartheta\kappa D_T N \\ &= \big(\vartheta\kappa\big)' N + \vartheta\kappa\big(-\vartheta\kappa T + \vartheta\tau B\big) \\ &= \big(\vartheta\kappa\big)' N - \big(\vartheta\kappa\big)^2 T + \big(\vartheta^2\kappa\tau\big) B \end{aligned}$$

olur. (3.2.4) bağıntısından bu ifadenin dik tümleyeni alınırsa

$$D_T^{\perp} H = \big(\vartheta\kappa\big)' N + \big(\vartheta^2\kappa\tau\big) B$$

bulunur. Normal konneksiyona göre tekrar türev alınırsa

$$D_T^{\perp} D_T^{\perp} H = D_T\big((\vartheta\kappa)' N + \vartheta^2\kappa\tau B\big)$$

$$= \big(\vartheta\kappa\big)''N + \big(\vartheta\kappa\big)'D_T N + \big(\vartheta^2\kappa\tau\big)'B + \vartheta^2\kappa\tau D_T B$$

$$= \big(\vartheta\kappa\big)''N + \big(\vartheta\kappa\big)'\big(-\vartheta\kappa T + \vartheta\tau B\big) + \big(\vartheta^2\kappa\tau\big)'B + \vartheta^2\kappa\tau\big(-\vartheta\tau N\big)$$

$$= \Big(\big(\vartheta\kappa\big)'' - \vartheta^3\big(\tau\big)^2\kappa\Big)N + \Big(\big(\vartheta\kappa\big)'\vartheta\tau + \big(\vartheta^2\kappa\tau\big)'\Big)B$$

olur ve buradan

$$\Delta^{\perp}H = \Big(\vartheta^3\tau^2\kappa - \big(\vartheta\kappa\big)''\Big)N - \Big(\big(\vartheta\kappa\big)'\vartheta\tau + \big(\vartheta^2\kappa\tau\big)'\Big)B$$

bulunur. $\Delta^{\perp}H = 0$ ise

$$\vartheta^3\tau^2\kappa - \big(\vartheta\kappa\big)'' = 0 \ \text{ ve } \ \big(\vartheta\kappa\big)'\vartheta\tau + \big(\vartheta^2\kappa\tau\big)' = 0$$

birinci önerme, $\Delta^{\perp}H = \lambda_0 H$ ise

$$\vartheta^3\tau^2\kappa - \big(\vartheta\kappa\big)'' = \lambda_0\kappa \ \text{ ve } \ \big(\vartheta\kappa\big)'\vartheta\tau + \big(\vartheta^2\kappa\tau\big)' = 0$$

ikinci önerme sağlanır. □

Lemma 4.0.3 Birim hızlı olmayan bir α eğrisinin T teğet vektörüne göre diferensiyel denklemi

$$D_T^3T - \Big(3\frac{\vartheta'}{\vartheta} + 2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\Big)D_T^2T$$

$$+\Big(\vartheta^2\big(\kappa^2+\tau^2\big) - \frac{\vartheta''}{\vartheta} - \frac{\kappa''}{\kappa} + \big(\frac{\vartheta'}{\vartheta} + \frac{\kappa'}{\kappa}\big)\big(3\frac{\vartheta'}{\vartheta} + \frac{\tau'}{\tau}\big) + 2\big(\frac{\kappa'}{\kappa}\big)^2\Big)D_T T + \Big(\vartheta^2\kappa\tau\big(\frac{\kappa}{\tau}\big)'\Big)T = 0$$

şeklinde verilir.

İspat. (3.1.4) bağıntısından N ve B vektörleri çekilirse

$$N = \frac{1}{\vartheta\kappa} D_T T \quad \text{ve} \quad B = \frac{1}{\vartheta\tau} D_T N + \frac{\kappa}{\tau} T$$

olur. B vektöründe, N nin yerine eşiti yazılır ve ardından türev alınırsa

$$\begin{aligned} B &= \frac{1}{\vartheta\tau} D_T \big(\frac{1}{\vartheta\kappa} D_T T\big) + \frac{\kappa}{\tau} T \\ &= \frac{1}{\vartheta\tau}\Big((\frac{1}{\vartheta\kappa})' D_T T + \frac{1}{\vartheta\tau}\frac{1}{\vartheta\kappa} D_T^2 T\Big) + \frac{\kappa}{\tau} T \\ &= \frac{1}{\vartheta\tau}(\frac{1}{\vartheta\kappa})' D_T T + \frac{1}{\vartheta^2\tau\kappa} D_T^2 T + \frac{\kappa}{\tau} T \end{aligned}$$

ve

$$D_T B = D_T\Big(\frac{1}{\vartheta\tau}(\frac{1}{\vartheta\kappa})' D_T T\Big) + D_T\Big(\frac{1}{\vartheta^2\kappa\tau} D_T^2 T\Big) + D_T\Big(\frac{\kappa}{\tau} T\Big)$$

olur. Diğer taraftan,

$$\begin{aligned} D_T B &= -\vartheta\tau N \\ &= -\vartheta\tau \frac{1}{\vartheta\kappa} D_T T \\ &= -\frac{\tau}{\kappa} D_T T \end{aligned}$$

dir. Bu ifade eşitliğin sol tarafında yerine yazılır ve türevler hesaplanırsa,

$$\begin{aligned} -\frac{\tau}{\kappa} D_T T &= D_T\Big(\frac{1}{\vartheta\tau}(\frac{1}{\vartheta\kappa})' D_T T\Big) + D_T\Big(\frac{1}{\vartheta^2\kappa\tau} D_T^2 T\Big) + D_T\Big(\frac{\kappa}{\tau} T\Big) \\ &= \big(\frac{1}{\vartheta\tau}(\frac{1}{\vartheta\kappa})'\big)' D_T T + \frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)' D_T^2 T + \big(\frac{1}{\vartheta^2\kappa\tau}\big)' D_T^2 T + \frac{1}{\vartheta^2\kappa\tau} D_T^3 T + \big(\frac{\kappa}{\tau}\big)' T \\ &\quad + \frac{\kappa}{\tau} D_T T \end{aligned}$$

$$\text{veya}\quad \Big(\frac{1}{\vartheta^2\kappa\tau}\Big)D_T^3T + \Big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)' + \big(\frac{1}{\vartheta^2\kappa\tau}\big)'\Big)D_T^2T$$

$$+ \Big(\frac{\kappa^2+\tau^2}{\kappa\tau} + \big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)'\big)'\Big)D_TT + \Big(\big(\frac{\kappa}{\tau}\big)'\Big)T = 0$$

bulunur. Burada D_T^2T ve D_TT nin katsayıları sırasıyla,

$$\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)' + \big(\frac{1}{\vartheta^2\kappa\tau}\big)' = -\frac{\vartheta'\kappa+\vartheta\kappa'}{\vartheta^3\tau\kappa^2} - \frac{2\vartheta\vartheta'\kappa\tau+\vartheta^2\kappa'\tau+\vartheta^2\kappa\tau'}{\vartheta^4\kappa^2\tau^2}$$

ve

$$\frac{\kappa}{\tau}+\frac{\tau}{\kappa}+\Big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)'\Big)' = \frac{\kappa^2+\tau^2}{\kappa\tau} - \frac{\vartheta''\kappa+2\vartheta'\kappa'+\vartheta\kappa''}{\vartheta^3\tau\kappa^2} + \frac{\big(3\vartheta^2\vartheta'\tau\kappa^2+\vartheta^3\tau'\kappa^2+2\vartheta^3\tau\kappa\kappa'\big)\big(\vartheta'\kappa+\vartheta\kappa'\big)}{\vartheta^6\tau^2\kappa^4}$$

dır. □

Lemma 4.0.4 Birim hızlı olmayan bir α eğrisinin N asli normal vektörüne veya B binormal vektörüne göre diferensiyel denklemi

$$D_T^\perp D_T^\perp N - \frac{\big(\vartheta\tau\big)'}{\vartheta\tau}D_T^\perp N + \big(\vartheta\tau\big)^2N = 0\,,$$

$$D_T^\perp D_T^\perp B - \frac{\big(\vartheta\tau\big)'}{\vartheta\tau}D_T^\perp B + \big(\vartheta\tau\big)^2B = 0$$

şeklinde verilir.

İspat. (3.2.3) bağıntısından N ve B vektörleri çekilirse

$$N = \frac{-1}{\vartheta\tau}D_T^\perp B \quad \text{ve} \quad B = \frac{1}{\vartheta\tau}D_T^\perp N$$

olur. $D_T^\perp N = \vartheta\tau B$ ifadesinin normal konneksiyona göre türevi alınırsa,

$$\begin{aligned} D_T^\perp\big(D_T^\perp N\big) &= D_T^\perp\big(\vartheta\tau B\big) \\ &= \big(\vartheta\tau\big)'B + \vartheta\tau D_T^\perp B \\ &= \big(\vartheta\tau\big)'\big(\frac{1}{\vartheta\tau}D_T^\perp N\big) + \vartheta\tau\big(-\vartheta\tau N\big) \end{aligned}$$

bulunur. Gerekli düzenlemeler yapılırsa,

$$D_T^{\perp} D_T^{\perp} N - \frac{\big(\vartheta\tau\big)'}{\vartheta\tau} D_T^{\perp} N + \big(\vartheta\tau\big)^2 N = 0$$

denklemi elde edilir. Benzer şekilde $D_T^{\perp} B = -\vartheta\tau N$ ifadesinin normal konneksiyona göre türevi alınırsa,

$$\begin{aligned} D_T^{\perp}\big(D_T^{\perp} B\big) &= D_T^{\perp}\big(-\vartheta\tau N\big) \\ &= \big(\vartheta\tau\big)' N - \vartheta\tau D_T^{\perp} N \\ &= \big(-\vartheta\tau\big)'\big(\frac{-1}{\vartheta\tau} D_T^{\perp} B\big) + \big(\vartheta\tau\big)^2 B \end{aligned}$$

olur. Gerekli düzenlemeler yapılırsa,

$$D_T^{\perp} D_T^{\perp} B - \frac{\big(\vartheta\tau\big)'}{\vartheta\tau} D_T^{\perp} B + \big(\vartheta\tau\big)^2 B = 0$$

denklemi elde edilir. □

Lemma 4.0.5 Birim hızlı olmayan bir α eğrisinin W Darboux vektörüne göre diferensiyel denklemi

$$a_1 D_T^3 W + a_2 D_T^2 W + a_3 D_T W + a_4 W = 0 \tag{4.0.1}$$

bağıntısıyla verilir. Burada a_1, a_2, a_3 ve a_4

$$\begin{aligned} a_1 &= \vartheta\big(\kappa\tau' - \kappa'\tau\big)^2, \quad a_2 = \Big(\vartheta\kappa''\tau - \vartheta\kappa\tau'' - \big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big)'\Big)\Big(\kappa\tau' - \kappa'\tau\Big), \\ a_3 &= \Big(\big(\kappa'''\tau - \kappa\tau''' + \vartheta^2(\kappa\tau' - \kappa'\tau)(\kappa^2 + \tau^2)\big)\big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big) \\ &\quad + \big(\vartheta\kappa''\tau - \vartheta\kappa\tau'' - (\vartheta\kappa\tau' - \vartheta\kappa'\tau)'\big)\big(\kappa''\tau - \kappa\tau''\big)\Big), \\ a_4 &= \Big(\big(\kappa'\tau''' - \kappa'''\tau' - \vartheta^2(\kappa\kappa' + \tau\tau')(\kappa\tau' - \kappa'\tau)\big)\big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big) \\ &\quad + \big(\vartheta\kappa''\tau - \vartheta\kappa\tau'' - (\vartheta\kappa\tau' - \vartheta\kappa'\tau)'(\kappa'\tau'' - \kappa''\tau')\big)\Big) \end{aligned}$$

şeklinde birer katsayıdır.

İspat. (3.1.5) bağıntısından $W = \tau T + \kappa B$ dir. Bu vektörün T yönünde türevleri

$$\begin{aligned}
D_T W &= D_T\big(\tau T + \kappa B\big) \\
D_T W &= \tau' T + \tau D_T T + \kappa' B + \kappa D_T B \\
&= \tau' T + \tau\vartheta\kappa N + \kappa' B - \kappa\vartheta\tau N \\
&= \tau' T + \kappa' B, \\
D_T^2 W &= D_T\big(\tau' T + \kappa' B\big) \\
&= \tau'' T + \tau' D_T T + \kappa'' B + \kappa' D_T B \\
&= \tau'' T + \big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big)N + \kappa'' B\,, \\
D_T^3 W &= D_T\Big(\tau'' T + \big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big)N + \kappa'' B\Big) \\
&= \tau''' T + \tau''\big(\vartheta\kappa N\big) + \big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big)' N \\
&\quad + \big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big)\big(-\vartheta\kappa T + \vartheta\tau B\big) + \kappa'' B - \kappa''\vartheta\tau N \\
&= \Big(\tau''' - \vartheta^2\kappa\big(\kappa\tau' - \kappa'\tau\big)\Big)T + \Big(\vartheta\kappa\tau'' - \vartheta\kappa''\tau + \big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big)'\Big)N \\
&\quad + \Big(\vartheta^2\tau\big(\kappa\tau' - \kappa'\tau\big) + \kappa'''\Big)B
\end{aligned}$$

bulunur. W ve D_TW ifadelerinden T ve B vektörleri çekilirse

$$T = \frac{\kappa}{\kappa\tau' - \kappa'\tau} D_TW - \frac{\kappa'}{\kappa\tau' - \kappa'\tau} W \quad \text{ve} \quad B = \frac{-\tau}{\kappa\tau' - \kappa'\tau} D_TW + \frac{\tau'}{\kappa\tau' - \kappa'\tau} W$$

olur. Bu vektörler D_T^2W türevinde yerine yazılırsa N vektörünün eşiti

$$N = \frac{1}{\vartheta(\kappa\tau' - \kappa'\tau)} D_T^2W + \frac{\kappa''\tau - \kappa\tau''}{\vartheta(\kappa\tau' - \kappa'\tau)^2} D_TW + \frac{\kappa'\tau'' - \kappa''\tau'}{\vartheta(\kappa\tau' - \kappa'\tau)^2} W$$

şeklinde bulunur. T, N ve B vektörleri D_T^3W ifadesinde yerine yazılırsa,

$$\begin{aligned} D_T^3W &= \left(\left(\vartheta\kappa\tau'' - \vartheta\kappa''\tau + \left(\vartheta\kappa\tau' - \vartheta\kappa'\tau\right)'\right) \frac{1}{\vartheta\kappa\tau' - \vartheta\kappa'\tau}\right) D_T^2W \\ &\quad + \left(\left(\kappa\tau''' - \kappa'''\tau - \vartheta^2\left(\kappa\tau' - \kappa'\tau\right)\left(\kappa^2 + \tau^2\right)\right) \frac{1}{\kappa\tau' - \kappa'\tau} \right. \\ &\quad + \left(\vartheta\kappa\tau'' - \vartheta\kappa''\tau + \left(\vartheta\kappa\tau' - \vartheta\kappa'\tau\right)'\right) \left. \frac{\kappa''\tau - \kappa\tau''}{\vartheta\left(\kappa\tau' - \kappa'\tau\right)^2} \right) D_TW \\ &\quad + \left(\left(\kappa'''\tau' - \kappa'\tau''' + \vartheta^2\left(\kappa\kappa' + \tau\tau'\right)\left(\kappa\tau' - \kappa'\tau\right)\right) \frac{1}{\kappa\tau' - \kappa'\tau} \right. \\ &\quad + \left(\vartheta\kappa\tau'' - \vartheta\kappa''\tau + \left(\vartheta\kappa\tau' - \vartheta\kappa'\tau\right)'\right) \left. \frac{\kappa'\tau'' - \kappa''\tau'}{\vartheta\left(\kappa\tau' - \kappa'\tau\right)^2} \right) W \end{aligned}$$

elde edilir. Bu eşitliğin her iki tarafı $\vartheta\left(\kappa\tau' - \kappa'\tau\right)^2$ ile çarpılır, ardından da D_T^3W, D_T^2W, D_TW ve W nin katsayılarına sırasıyla a_1, a_2, a_3 ve a_4 denilirse, ispat tamamlanır. $\square$

Lemma 4.0.6 Birim hızlı olmayan bir α eğrisinin C birim Darboux vektörüne göre diferensiyel denklemi

$$D_T^3 C + \sigma_1 D_T^2 C + \sigma_2 D_T C + \sigma_3 C \;=\; 0 \tag{4.0.2}$$

bağıntısıyla verilir. Burada σ_1, σ_2 ve σ_3

$$\sigma_1 \;=\; -\frac{\phi''}{\phi'} - \frac{\big(\phi'\vartheta \parallel W \parallel\big)'}{\vartheta \parallel W \parallel \phi'}\,,$$

$$\sigma_2 \;=\; \big(\vartheta \parallel W \parallel\big)^2 + (\phi')^2 - \big(\frac{\phi''}{\phi'}\big)' + \frac{\big(\phi'\vartheta \parallel W \parallel\big)'}{\vartheta \parallel W \parallel \big(\phi'\big)^2}\phi''\,,$$

$$\sigma_3 \;=\; \big((\phi')^2\big)' - \frac{\big(\phi'\vartheta \parallel W \parallel\big)'}{\vartheta \parallel W \parallel}\phi'$$

şeklinde birer katsayıdır.

İspat. (3.1.6) bağıntısından C birim Darboux vektörü

$$C \;=\; sin\phi T + cos\phi B$$

dir. Bu vektörün T yönünde birinci türevi

$$\begin{aligned} D_T C &= D_T\big(sin\phi T + cos\phi B\big) \\ &= \phi' cos\phi T + sin\phi \vartheta\kappa N - \phi' sin\phi B - cos\phi\vartheta\tau N \\ &= \phi'\big(cos\phi T - sin\phi B\big) \end{aligned}$$

olur. C ve $D_T C$ ifadeledinden T ve B vektörleri çekilirse,

$$T = sin\phi C + \frac{cos\phi}{\phi'} D_T C \quad \text{ve} \quad B = cos\phi C - \frac{sin\phi}{\phi'} D_T C$$

bulunur. Bu vektörler C nin ikinci türevinde yerine yazılırsa,

$$
\begin{aligned}
D_T^2C &= D_T\big(\phi'(cos\phi T - sin\phi B)\big) \\
&= \phi''\big(cos\phi T - sin\phi B\big) - \big(\phi'\big)^2\big(sin\phi T + cos\phi B\big) + \phi'\vartheta\big(cos\phi\kappa + sin\phi\tau\big)N \\
&= \phi'' cos\phi\big(sin\phi C + \frac{cos\phi}{\phi'}D_TC\big) - \phi'' sin\phi\big(cos\phi C - \frac{sin\phi}{\phi'}D_TC\big) \\
&\quad + \phi'\vartheta \parallel W \parallel N - \big(\phi'\big)^2 \\
&= \frac{\phi''}{\phi'}D_TC - \big(\phi'\big)^2 C + \phi'\vartheta \parallel W \parallel N
\end{aligned}
$$

olur. Bu eşitlikten N vektörü çekilirse,

$$
N = \frac{1}{\vartheta\big(\phi'\big)^2 \parallel W \parallel}\Big(\phi' D_T^2C - \phi'' D_TC + (\phi')^3 C\Big)
$$

bulunur. C nin T yönünde üçüncü türevi alınırsa

$$
\begin{aligned}
D_T^3C &= D_T\Big(\frac{\phi''}{\phi'}D_TC - \big(\phi'\big)^2 C + \phi'\vartheta \parallel W \parallel N\Big) \\
&= \big(\frac{\phi''}{\phi'}\big)' D_TC + \frac{\phi''}{\phi'}D_T^2C - \Big(\big(\phi'\big)^2\Big)' C - \big(\phi'\big)^2 D_TC + \vartheta^2\phi' \parallel W \parallel \big(-\kappa T + \tau B\big) \\
&\quad + \big(\vartheta\phi' \parallel W \parallel\big)' \frac{1}{\vartheta\big(\phi'\big)^2 \parallel W \parallel}\Big(\phi' D_T^2C - \phi'' D_TC + \big(\phi'\big)^3 C\Big)
\end{aligned}
$$

$$= \Big(\frac{\phi''}{\phi'} + \frac{\big(\phi'\vartheta \parallel W \parallel\big)'}{\vartheta \parallel W \parallel \phi'}\Big) D_T^2 C$$

$$+\Big(\big(\frac{\phi''}{\phi'}\big)' - \big(\vartheta \parallel W \parallel\big)^2 - \big(\phi'\big)^2 - \frac{\big(\phi'\vartheta \parallel W \parallel\big)'}{\vartheta \parallel W \parallel (\phi')^2}\phi''\Big) D_T C$$

$$+\Big(\frac{\big(\phi'\vartheta \parallel W \parallel\big)'}{\vartheta \parallel W \parallel}\phi' - \big((\phi')^2\big)'\Big) C$$

bulunur. Bu denklemdeki $D_T^2 C$, $D_T C$ ve C nin katsayılarına sırasıyla σ_1, σ_2 ve σ_3 denilirse, ispat tamamlanır. □

Lemma 4.0.7 Birim hızlı olmayan bir α eğrisinin $W^\perp$ normal Darboux vektörüne göre diferensiyel denklemi

$$\epsilon_2 \big(D_T^\perp\big)^2 W^\perp + \epsilon_1 D_T^\perp W^\perp + \epsilon_0 W^\perp = 0 \qquad (4.0.3)$$

bağıntısıyla verilir. Burada ϵ_0, ϵ_1, ϵ_2

$$\epsilon_0 = \kappa'\Big(\big(\vartheta\kappa\tau\big)' + \vartheta\kappa'\tau\Big) - \vartheta\kappa\tau\Big(\kappa'' - \vartheta^2\kappa\tau^2\Big),$$

$$\epsilon_1 = -\kappa\Big(\big(\vartheta\kappa\tau\big)' + \vartheta\kappa'\tau\Big),$$

$$\epsilon_2 = \vartheta\kappa^2\tau$$

şeklinde birer katsayıdır.

İspat. (3.2.3) bağıntısından $W^\perp = \kappa B$ dir. Bu vektörün T yönünde normal konneksiyona göre türevi alınırsa,

$$D_T^\perp W^\perp = -\vartheta\kappa\tau N + \kappa' B$$

olur. $W^\perp$ ve $D_T^\perp W^\perp$ ifadelerinden N vektörü çekilirse

$$N = \big(\frac{-1}{\vartheta\kappa\tau}\big) D_T^\perp W^\perp + \frac{\kappa'}{\vartheta\kappa^2\tau} W^\perp \quad \text{ve} \quad B = \big(\frac{1}{\kappa}\big) W^\perp$$

bulunur. $W^{\perp}$ vektörünün normal konneksiyona göre ikinci türevi alınırsa

$$\begin{aligned}\big(D_T^{\perp}\big)^2 W^{\perp} &= D_T^{\perp}\big(-\vartheta\kappa\tau N+\kappa' B\big)\\ &= -\Big(\big(\vartheta\kappa\tau\big)'+\vartheta\kappa'\tau\Big)N+\Big(\kappa''-\vartheta^2\kappa\tau^2\Big)B\end{aligned}$$

olur. N ve B vektörlerinin eşitleri $\big(D_T^{\perp}\big)^2 W^{\perp}$ ifadesinde yerine yazılırsa

$$\vartheta\kappa^2\tau\big(D_T^{\perp}\big)^2 W^{\perp}-\kappa\Big((\vartheta\kappa\tau)'+\vartheta\kappa'\tau\Big)D_T^{\perp}W^{\perp}+\Big(\kappa'((\vartheta\kappa\tau)'+\vartheta\kappa'\tau)-\vartheta\kappa\tau(\kappa''-\vartheta^2\kappa\tau^2)\Big)W^{\perp}=0$$

elde edilir. □

Lemma 4.0.8 Birim hızlı olmayan bir α eğrisinin $C^{\perp}$ birim normal Darboux vektörüne göre diferensiyel denklemi

$$\varsigma_3\big(D_T^{\perp}\big)^2 C^{\perp}+\varsigma_2 D_T^{\perp}C^{\perp}+\varsigma_1 C^{\perp}=0 \tag{4.0.4}$$

bağıntısıyla verilir. Burada ς_1, ς_2 ve ς_3

$$\begin{aligned}\varsigma_1 &= \phi' sin\phi(\phi' sin\phi\vartheta\tau-(\vartheta\tau cos\phi)')+\vartheta\tau cos\phi(\vartheta^2\tau^2 cos\phi+(\phi' sin\phi)'),\\ \varsigma_2 &= cos\phi(\phi' sin\phi\vartheta\tau-(\vartheta\tau cos\phi)'),\\ \varsigma_3 &= \vartheta\tau cos^2\phi\end{aligned}$$

şeklinde birer katsayıdır.

İspat. (3.2.3) bağıntısından $C^{\perp}=cos\phi B$ dir. Bu vektörün T yönünde normal konneksiyona göre türevi alınırsa

$$D_T^{\perp}C^{\perp} = -\vartheta\tau cos\phi N-\phi' sin\phi B$$

olur. $C^{\perp}$ ve $D_T^{\perp}C^{\perp}$ ifadelerinden N ve B vektörleri çekilirse

$$N=(\frac{-1}{\vartheta\tau cos\phi})D_T^{\perp}C^{\perp}-\frac{\phi' sin\phi}{\vartheta\tau cos^2\phi}C^{\perp}\quad \text{ve}\quad B=(\frac{1}{cos\phi})C^{\perp}$$

bulunur. $C^{\perp}$ vektörünün T yönünde normal konneksiyona göre ikinci türevi

$$\begin{aligned}\big(D_T^{\perp}\big)^2C^{\perp} &= D_T^{\perp}\big(-\vartheta\tau cos\phi N-\phi' sin\phi B\big)\\ &= \big(-\vartheta\tau cos\phi\big)'N-\vartheta\tau cos\phi\big(-\vartheta\kappa T+\vartheta\tau B\big)-\big(\phi' sin\phi\big)'B+\big(\phi' sin\phi\big)\big(\vartheta\tau N\big)\\ &= \Big(\phi' sin\phi\vartheta\tau-\big(\vartheta\tau cos\phi\big)'\Big)N-\Big(\vartheta^2\tau^2 cos\phi+\big(\phi' sin\phi\big)'\Big)B\end{aligned}$$

dir. Bu türevde N ve B nin eşitleri yerine yazılırsa

$$\begin{aligned}&\vartheta\tau cos^2\phi\big(D_T^{\perp}\big)^2C^{\perp}+cos\phi\Big(\phi' sin\phi\vartheta\tau-\big(\vartheta\tau cos\phi\big)'\Big)D_T^{\perp}C^{\perp}\\ &+\Big(\phi' sin\phi\big(\phi' sin\phi\vartheta\tau-(\vartheta\tau cos\phi)'\big)+\vartheta\tau cos\phi\big(\vartheta^2\tau^2 cos\phi+(\phi' sin\phi)'\big)\Big)C^{\perp}=0\end{aligned}$$

elde edilir. Böylece ispat tamamlanır. □

Lemma 4.0.9 Birim hızlı olmayan bir α eğrisinin W Darboux vektörüne göre biharmonik olma ve 1. tipten harmonik olma koşulları aşağıdaki gibi verilir:

1) α eğrisi Darboux vektörüne göre biharmoniktir ancak ve ancak

$$\tau''=0,\quad \vartheta\kappa\tau'-\vartheta\kappa'\tau=0\,,\quad \kappa''=0.$$

2) α eğrisi Darboux vektörüne göre 1.tipten harmoniktir ancak ve ancak

$$\tau''=-\lambda_0\tau,\quad \vartheta\kappa\tau'-\vartheta\kappa'\tau=0\,,\quad \kappa''=-\lambda_0\kappa\,,\quad \lambda_0\in\mathbb{R}.$$

İspat. Teorem 3.2.7 den W Darboux vektörünün Laplace görüntüsü

$$\begin{aligned}\Delta W &= -D_T^2 W \\ &= -\tau'' T + \big(\vartheta\kappa'\tau - \vartheta\kappa\tau'\big)N - \kappa'' B\end{aligned}$$

bulunur. $\Delta W = 0$, için birinci önerme ve $\Delta W = \lambda_0 W$ için ikinci önerme sağlanır. $\square$

Lemma 4.0.10 Birim hızlı olmayan bir α eğrisinin C birim Darboux vektörüne göre biharmonik olma ve 1. tipten harmonik olma koşulları aşağıdaki gibi verilir:
1) α eğrisi C birim Darboux vektörüne göre biharmoniktir ancak ve ancak

$$\phi'' cos\phi - (\phi')^2 sin\phi = 0, \quad \phi'' sin\phi + (\phi')^2 cos\phi = 0\,, \quad \phi'\vartheta \parallel W \parallel = 0$$

2) α eğrisi C birim Darboux vektörüne göre 1.tipten harmoniktir ancak ve ancak

$$\phi'' cos\phi - (\phi')^2 sin\phi = -\lambda sin\phi, \;\; \phi'' sin\phi + (\phi')^2 cos\phi = 0\,, \;\; \phi'\vartheta \parallel W \parallel = -\lambda_0 cos\phi\,, \;\; \lambda_0 \in \mathbb{R}.$$

İspat. Teorem 3.2.7 den C birim Darboux vektörünün Laplace görüntüsü

$$\begin{aligned}\Delta C &= -D_T^2 C \\ &= \big((\phi')^2 sin\phi - \phi'' cos\phi\big)T + \big(\phi'' sin\phi + (\phi')^2 cos\phi\big)N - \big(\phi'\vartheta \parallel W \parallel\big)B\end{aligned}$$

bulunur. $\Delta C = 0$ ise birinci önerme, $\Delta C = \lambda_0 C$ ise ikinci önerme sağlanır. $\square$

Lemma 4.0.11 Birim hızlı olmayan bir α eğrisinin $W^{\perp}$ normal Darboux vektörüne göre, zayıf biharmonik olma ve 1. tipten harmonik olma koşulları aşağıdaki gibi verilir:
1. α eğrisi normal Darboux vektörüne göre zayıf biharmoniktir ancak ve ancak

$$(\vartheta\kappa\tau)' + \vartheta\kappa'\tau = 0 \text{ ve } \kappa'' - \vartheta^2\kappa\tau^2 = 0.$$

2. α eğrisi normal Darboux vektörüne göre 1.tipten harmoniktir ancak ve ancak

$$(\vartheta\kappa\tau)' + \vartheta\kappa'\tau = 0 \quad \text{ve} \quad \vartheta^2\kappa\tau^2 - \kappa'' = \lambda_0\kappa\,, \quad \lambda_0 \in \mathbb{R}.$$

İspat. Teorem 3.2.8 den $W^{\perp}$ vektörünün normal Laplace görüntüsü

$$\begin{aligned}(\Delta)^{\perp}W^{\perp} &= -D_T^2 W^{\perp} \\ &= \big((\vartheta\kappa\tau)' + \vartheta\kappa'\tau\big)N - \big(\kappa'' - \vartheta^2\kappa\tau^2\big)B\end{aligned}$$

bulunur. $(\Delta)^{\perp}W^{\perp} = 0$ ise birinci önerme, $(\Delta)^{\perp}W^{\perp} = \lambda_0 W^{\perp}$ ise ikinci önerme sağlanır. Bu da ispatı tamamlar. □

Lemma 4.0.12 Birim hızlı olmayan bir α eğrisinin $C^{\perp}$ birim normal Darboux vektörüne göre, zayıf biharmonik olma ve 1. tipten harmonik olma koşulları aşağıdaki gibi verilir:
1. α eğrisi birim normal Darboux vektörüne göre zayıf biharmoniktir ancak ve ancak

$$\phi' sin\phi\vartheta\tau - (\vartheta\tau cos\phi)' = 0 \text{ ve } \vartheta^2\tau^2 cos\phi + (\phi' sin\phi)' = 0.$$

2. α eğrisi birim normal Darboux vektörüne göre 1.tipten harmoniktir ancak ve ancak

$$\phi' sin\phi\vartheta\tau - (\vartheta\tau cos\phi)' = 0 \quad \text{ve} \quad \vartheta^2\tau^2 cos\phi + (\phi' sin\phi)' = \lambda_0 cos\phi, \quad \lambda_0 \in \mathbb{R}.$$

İspat. Teorem 3.2.8 den $C^{\perp}$ vektörünün normal Laplace görüntüsü

$$\begin{aligned}\Delta)^{\perp}C^{\perp} &= -(D_N^{\perp})^2 C^{\perp} \\ &= \big((\vartheta\tau cos\phi)' - \phi' sin\phi\vartheta\tau\big)N + \big(\vartheta^2\tau^2 cos\phi + (\phi' sin\phi)'\big)B\end{aligned}$$

bulunur. $(\Delta)^{\perp}C^{\perp} = 0$ için 1. önerme, $(\Delta)^{\perp}C^{\perp} = \lambda_0 C^{\perp}$ için 2. önerme sağlanır. □

4.1 İnvolüt Eğrisinin Karakterizasyonları

4.1.1 İnvolüt Eğrisinin Konneksiyona Göre Frenet Formülleri

Bu kısımda α eğrisinin Frenet aparatları $\{T\,, N\,, B, \kappa, \tau\}$ ve β eğrisinin Frenet aparatları $\{T_\beta,\ N_\beta,\ B_\beta,\ \kappa_\beta,\ \tau_\beta\}$ ile gösterilecektir.

Teorem 4.1.1 β eğrisi α nın bir involütü olsun. α eğrisinin Frenet vektörlerinin asli normal vektörü N yönünde türevleri

$$D_N T = \kappa N, \qquad D_N N = -\kappa T + \tau B, \qquad D_N B = -\tau N \tag{4.1.1}$$

bağıntısıyla verilir.

İspat. (3.1.4) bağıntısından $D_{T_\beta} T_\beta = \vartheta \kappa_\beta N_\beta$ dir. Bu eşitliğin sağ tarafı (3.1.7) ve (3.1.8) bağıntılarından

$$\begin{aligned}
D_{T_\beta} T_\beta &= \vartheta \kappa_\beta N_\beta \\
&= \vartheta \frac{\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}\Big(\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}} T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}} B\Big) \\
&= -\kappa T + \tau B
\end{aligned}$$

olur. Aynı bağıntılardan $T_\beta = N$ ve dolayısıyla

$$\begin{aligned}
D_{T_\beta} T_\beta &= D_N N \\
&= -\kappa T + \tau B
\end{aligned}$$

bulunur.

Benzer şekilde (3.1.7) ve (3.1.8) bağıntılarından $D_{T_\beta}N_\beta$ vektörü

$$\begin{aligned}
D_{T_\beta}N_\beta &= -\vartheta\kappa_\beta T_\beta + \vartheta\tau_\beta B_\beta \\
&= \vartheta\Big(\frac{-\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}\Big)N + \vartheta\Big(\frac{\kappa\tau'-\kappa'\tau}{(c-s)\kappa(\kappa^2+\tau^2)}(\frac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}})\Big) \\
&= \Big(\frac{(\kappa\tau'-\kappa'\tau)\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)T - \Big(\sqrt{\kappa^2+\tau^2}\Big)N + \Big(\frac{(\kappa\tau'-\kappa'\tau)\kappa}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)B \qquad (4.1.2)
\end{aligned}$$

olur. Diğer yandan bu eşitliğin sol tarafı

$$\begin{aligned}
D_{T_\beta}N_\beta &= D_N\Big(\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B\Big) \\
&= -(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})'T - \frac{\kappa.D_N T}{\sqrt{\kappa^2+\tau^2}} + (\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'B + \frac{\tau.D_N B}{\sqrt{\kappa^2+\tau^2}} \qquad (4.1.3)
\end{aligned}$$

olur. (4.1.2) ve (4.1.3) ten

$$\begin{aligned}
-\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}D_N T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}D_N B &= \Big(\frac{(\kappa\tau'-\kappa'\tau)\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}} + (\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})'\Big)T - \Big(\sqrt{\kappa^2+\tau^2}\Big)N \\
&\quad + \Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)\kappa}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}} - (\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'\Big)B \qquad (4.1.4)
\end{aligned}$$

bulunur. Benzer şekilde $D_{T_\beta}B_\beta$ vektörü (3.1.7) ve (3.1.8) bağıntılarından

$$\begin{aligned}
D_{T_\beta}B_\beta &= -\vartheta\tau_\beta N_\beta \\
&= \vartheta\Big(-\frac{\kappa\tau'-\kappa'\tau}{(c-s)\kappa\big(\kappa^2+\tau^2\big)}(\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B)\Big) \\
&= \Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)\kappa}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}\Big)T - \Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)\tau}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}\Big)B \qquad (4.1.5)
\end{aligned}$$

olur. Diğer taraftan eşitliğin sol tarafı

$$\begin{aligned} D_{T_\beta}B_\beta &= D_N\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}T+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B\Big) \\ &= \Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'T+\frac{\tau\,D_NT}{\sqrt{\kappa^2+\tau^2}}+\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'B+\frac{\kappa\,D_NB}{\sqrt{\kappa^2+\tau^2}} \qquad (4.1.6) \end{aligned}$$

bulunur. Böylece (4.1.5) ve (4.1.6) bağıntılarından

$$\begin{aligned} \frac{\tau}{\sqrt{\kappa^2+\tau^2}}D_NT+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}D_NB &= \Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)\kappa}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}-\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\Big)T \\ &\quad -\Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)\tau}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}+\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'\Big)B \qquad (4.1.7) \end{aligned}$$

olur. (4.1.4) ve (4.1.7) bağıntılarından D_NT ve D_NB hesaplanırsa

$$\begin{aligned} D_NT &= \frac{-1}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'+\tau\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\Big)T+\kappa N \\ &\quad +\frac{1}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'-\tau\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'-\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big)B \\ &= \kappa N \qquad (4.1.8) \end{aligned}$$

ve

$$\begin{aligned} D_NB &= \frac{1}{\sqrt{\kappa^2+\tau^2}}\Big(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}+\tau\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'-\kappa\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\Big)T \\ &\quad -\big(\tau\big)N-\frac{1}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'+\tau\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\Big)B \\ &= -\tau N \qquad (4.1.9) \end{aligned}$$

şeklinde elde edilir. □

Teorem 4.1.2 β eğrisi α nın bir involütü olsun. Konneksiyona göre β involüt eğrisinin harmoniklik koşulları α eğrisinin Frenet aparatları türünden aşağıdaki önermelerle verilir:
1. β involüt eğrisi biharmoniktir ancak ve ancak

$$\kappa'' - \kappa^3 - \kappa\tau^2 = 0, \quad \kappa\kappa' + \tau\tau' = 0, \quad \kappa^2\tau + \tau^3 - \tau'' = 0.$$

2. β involüt eğrisi 1. tipten harmoniktir ancak ve ancak

$$\kappa'' - \kappa^3 - \kappa\tau^2 = -\lambda_0\kappa, \quad \kappa\kappa' + \tau\tau' = 0, \quad \kappa^2\tau + \tau^3 - \tau'' = \lambda_0\tau, \ \ \lambda_0 \in \mathbb{R}.$$

İspat. (4.1.1) bağıntısından H_β ortalama eğrilik vektörünün yeni ifadesi

$$H_\beta = D_N N = -\kappa T + \tau B$$

olur. (3.2.2) bağıntısından H_β nin Laplace görüntüsü

$$\begin{aligned}
\Delta H_\beta &= -D_N^2\big(H_\beta\big) \\
&= -D_N^2\big(D_N N\big) \\
&= -D_N^2\big(-\kappa T + \tau B\big) \\
&= -D_N D_N\big(-\kappa T + \tau B\big) \\
&= -D_N\big(-\kappa' T - \kappa D_N T + \tau' B + \tau D_N B\big) \\
&= -D_N\big(-\kappa' T - \kappa\kappa N + \tau' B + \tau(-\tau N)\big) \\
&= -\Big(-\kappa'' T - \kappa' D_N T - \big(\kappa^2 + \tau^2\big)' N - \big(\kappa^2 + \tau^2\big) D_N N + \tau'' B + \tau' D_N B\Big) \\
&= \Big(\kappa'' T + \kappa'\kappa N + \big(\kappa^2 + \tau^2\big)' N + \big(\kappa^2 + \tau^2\big)\big(-\kappa T + \tau B\big) - \tau'' B + \tau'\tau N\Big) \\
&= \big(\kappa'' - \kappa^3 - \kappa\tau^2\big) T + 3\big(\kappa\kappa' + \tau\tau'\big) N + \big(\kappa^2\tau + \tau^3 - \tau''\big) B
\end{aligned}$$

bulunur. $\Delta H_\beta = 0$ ise birinci önerme, $\Delta H_\beta = \lambda_0 H_\beta$ ise ikinci önerme sağlanır. $\square$

Sonuç 4.1.1 β eğrisi α nın bir involütü olsun. α eğrisi dairesel helis ise, involüt eğrisi β konneksiyona göre 1.tipten harmonik bir eğridir.

İspat. α eğrisi dairesel helis olsun. O zaman $\kappa = sbt$ ve $\tau = sbt$ tir. (4.1.1) den ortalama eğrilik vektörü $H_\beta = -\kappa T + \tau B$ ve ortalama eğriliğin Laplace görüntüsü de $\Delta H = (\kappa^2 + \tau^2)(-\kappa T + \tau B)$ olur. Buradan $\Delta H_\beta = (\kappa^2 + \tau^2) H_\beta$ elde edilir ve $\lambda_0 = \kappa^2 + \tau^2$ olup, Teorem 4.1.2 de ikinci önermenin sağlandığı görülür. $\square$

Sonuç 4.1.2 β eğrisi α nın bir involütü olsun. α eğrisi genel helis ise, involüt eğrisi β konneksiyona göre biharmoniktir.

İspat. α eğrisi genel helis olsun. O zaman $\tau/\kappa = sbt$ tir. Bu eşitliğin her iki tarafı türevlenirse, $\kappa\tau' - \kappa'\tau = 0$ olur. Buradan da $\tau/\kappa = \tau'/\kappa' = sbt$ bulunur. Tekrar türev alınırsa, $\tau/\kappa = \tau''/\kappa'' = sbt$ elde edilir ve Teorem 4.1.2 de birinci önerme sağlanır. $\square$

Örnek 4.1.1 $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ verilsin. α eğrisinin Frenet vektörleri ve eğrilikleri
$T = \frac{1}{\sqrt{2}}(-sins, coss, 1)$, $N = (-coss, -sins, 0)$, $B = \frac{1}{\sqrt{2}}(sins, -coss, 1)$
$\kappa = \frac{1}{\sqrt{2}}$ ve $\tau = \frac{1}{\sqrt{2}}$ dir. α eğrisinin Frenet formülleri

$$D_T T = \frac{-1}{\sqrt{2}}(coss, sins, 0), \quad D_T N = (sins, -coss, 0), \quad D_T B = \frac{1}{\sqrt{2}}(coss, sins, 0)$$

dir. Buna göre β involüt eğrisi

$$\beta(s) = \frac{1}{\sqrt{2}}\big(coss - (c-s)sins,\ sins + (c-s)coss,\ c\big)$$

olur. β eğrisinin Frenet vektörleri ve eğrilikleri

$$T_\beta = (-coss, -sins, 0), \quad N_\beta = (sins, -coss, 0), \quad B_\beta = (0,0,1)$$

$$\kappa_\beta = \frac{\| \beta'(s) \wedge \beta''(s) \|}{\| \beta'(s) \|^3} = \frac{\sqrt{2}}{| c-s |}, \qquad \tau_\beta = \frac{det(\beta'(s), \beta''(s), \beta'''(s))}{\| \beta'(s) \wedge \beta''(s) \|^2} = 0$$

şeklinde bulunur. β eğrisinin H_β ortalama eğrilik vektörünün Laplace görüntüsü

$$\begin{aligned}\Delta H_\beta &= \Big(3\big(\vartheta\kappa_\beta\big)'\vartheta\kappa_\beta\Big)T_\beta + \Big(\big(\vartheta\kappa_\beta\big)^3 + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)^2 - \big(\vartheta\kappa_\beta\big)''\Big)N_\beta \\ &\quad -\Big(2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)B_\beta\end{aligned}$$

olur. Burada

$$\vartheta = \frac{\mid c-s\mid}{\sqrt{2}}, \quad \kappa_\beta = \frac{\sqrt{2}}{\mid c-s\mid} \ \text{ ve } \ \tau_\beta = 0$$

ifadeleri ΔH_β de yerlerine yazılırsa, $\Delta H_\beta = H_\beta$ elde edilir. Bu durumda β eğrisi 1.tipten harmoniktir.

Diğer taraftan β eğrisinin harmonikliği (4.1.1) bağıntısına göre hesaplanırsa,

$$H_\beta = -\kappa T + \tau B \quad \text{ve} \quad \Delta H_\beta = -D_N^2\big(-\kappa T + \tau B\big)$$

olur. Burada N yönündeki türevler

$$D_N T = \frac{1}{\sqrt{2}}N, \qquad D_N N = -\frac{1}{\sqrt{2}}T + \frac{1}{\sqrt{2}}B, \qquad D_N B = -\frac{1}{\sqrt{2}}N$$

bulunur. Ortalama eğriliğin Laplace görüntüsü

$$\begin{aligned}\Delta H_\beta &= -D_{T_\beta}^2\big(\vartheta\kappa_\beta N_\beta\big) \\ &= -D_N^2\big(-\kappa T + \tau B\big) \\ &= -D_N\Big(\frac{-1}{\sqrt{2}}D_N T + \frac{1}{\sqrt{2}}D_N B\Big) \\ &= -D_N\big(-N\big) \\ &= D_N N\end{aligned}$$

dir. Böylece $\Delta D_N N = D_N N$ eşitliğinden β nın 1. tipten harmonik olduğu görülür.

4.1.2 İnvolüt Eğrisinin Normal Konneksiyona Göre Frenet Formülleri

Teorem 4.1.3 β eğrisi α nın bir involütü olsun. Normal konneksiyona göre α eğrisinin N asli normal yönünde türevleri

$$D_N^{\perp} T = 0\,, \qquad D_N^{\perp} B = 0 \tag{4.1.10}$$

bağıntısıyla verilir.

İspat. (3.2.3) bağıntısından $D_{T_\beta}^{\perp} N_\beta = \vartheta \tau_\beta B_\beta$ ve $D_{T_\beta}^{\perp} B_\beta = -\vartheta \tau_\beta N_\beta$ yazılır.

(3.1.7) ve (3.1.8) bağıntılarından $D_{T_\beta}^{\perp} N_\beta = \vartheta \tau_\beta B_\beta$ ifadesinin sol tarafı

$$\begin{aligned} D_{T_\beta}^{\perp} N_\beta &= D_N^{\perp}\Big(\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}} T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}} B\Big) \\ &= \big(\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}\big)' T + \big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' B - \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} D_N^{\perp} T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}} D_N^{\perp} B, \end{aligned} \tag{4.1.11}$$

sağ tarafı

$$\vartheta \tau_\beta B_\beta = \mid c-s \mid \kappa \frac{\kappa\tau' - \kappa'\tau}{\mid c-s \mid \kappa\big(\kappa^2+\tau^2\big)} \Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} B\Big) \tag{4.1.12}$$

olur. Benzer şekilde $D_{T_\beta}^{\perp} B_\beta = -\vartheta \tau_\beta N_\beta$ ifadesinin sol tarafı

$$\begin{aligned} D_{T_\beta}^{\perp} B_\beta &= D_N^{\perp}\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} B\Big) \\ &= \big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' T + \big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)' B + \frac{\tau}{\sqrt{\kappa^2+\tau^2}} D_N^{\perp} T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} D_N^{\perp} B, \end{aligned} \tag{4.1.13}$$

sağ tarafı

$$-\vartheta \tau_\beta N_\beta = \mid s-c \mid \kappa \frac{\kappa\tau' - \kappa'\tau}{\mid c-s \mid \kappa\big(\kappa^2+\tau^2\big)} \Big(\frac{-\kappa T + \tau B}{\sqrt{\kappa^2+\tau^2}}\Big) \tag{4.1.14}$$

şeklinde bulunur.

(4.1.11), (4.1.12), (4.1.13) ve (4.1.14) bağıntılarından,

$$\begin{aligned}\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}D_N^{\perp}T+\frac{\tau}{\sqrt{\kappa^2+\tau^2}}D_N^{\perp}B &= \Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)\tau}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}+\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)T \\ &\quad +\Big(\frac{\kappa\big(\kappa\tau'-\kappa'\tau\big)}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}-\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)B \end{aligned} \tag{4.1.15}$$

ve

$$\begin{aligned}\frac{\tau}{\sqrt{\kappa^2+\tau^2}}D_N^{\perp}T+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}D_N^{\perp}B &= \Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)\kappa}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}-\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)T \\ &\quad -\Big(\frac{\kappa\big(\kappa\tau'-\kappa'\tau\big)}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}+\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)B \end{aligned} \tag{4.1.16}$$

ifadeleri elde edilir. Bu iki eşitlikten gerekli işlemler yapılırsa, $D_N^{\perp}T$ ve $D_N^{\perp}B$ nin eşitleri elde edilmiş olur. □

4.1.3 İnvolüt Eğrisinin Konneksiyona Göre Diferensiyel Denklemleri

Teorem 4.1.4 β eğrisi α nın bir involütü olsun. β involüt eğrisinin konneksiyona göre diferensiyel denklemi

$$n_3D_N^3N+n_2D_N^2N+n_1D_NN+n_0N=0$$

bağıntısıyla verilir. Burada $n_0,\ n_1,\ n_2$ ve n_3

$$\begin{aligned} n_0 &= \Big(3\big(\kappa\kappa'+\tau\tau'\big)\big(\kappa\tau'-\kappa'\tau\big)+\big(\kappa^2+\tau^2\big)\big(\kappa''\tau-\kappa\tau''\big)\Big), \\ n_1 &= \Big(\kappa'\tau''-\kappa''\tau'+\big(\kappa^2+\tau^2\big)\big(\kappa\tau'-\kappa'\tau\big)\Big), \\ n_2 &= \kappa''\tau-\kappa\tau'' \end{aligned}$$

ve $n_3 = \kappa\tau' - \kappa'\tau$

şeklinde birer katsayıdır.

İspat. (4.1.1) bağıntısından $D_N N = -\kappa T + \tau B$ yazılır. Bu ifadenin N yönünde birinci ve ikinci türevi alınırsa,

$$
\begin{aligned}
D_N^2 N &= D_N\big(D_N N\big) \\
&= D_N\big(-\kappa T + \tau B\big) \\
&= -\kappa' T - \kappa D_N T + \tau' B + \tau D_N B \\
&= -\kappa' T - \kappa^2 N + \tau' B - \tau^2 N \\
&= -\kappa' T - \big(\kappa^2 + \tau^2\big) N + \tau' B, \\
D_N^3 N &= D_N\big(D_N^2 N\big) \\
&= D_N\Big(-\kappa' T - \big(\kappa^2 + \tau^2\big) N + \tau' B\Big) \\
&= -\kappa'' T - \kappa' D_N T - \big(\kappa^2 + \tau^2\big)' N + -\big(\kappa^2 + \tau^2\big) D_N N + \tau'' B + \tau' D_N B \\
&= -\kappa'' T - 3\big(\kappa\kappa' + \tau\tau'\big) N + \big(\kappa^2 + \tau^2\big)\kappa T - \big(\kappa^2 + \tau^2\big)\tau B + \tau'' B \\
&= \Big(\kappa\big(\kappa^2 + \tau^2\big) - \kappa''\Big) T - 3\Big(\kappa\kappa' + \tau\tau'\Big) N + \Big(\tau'' - (\kappa^2 + \tau^2)\tau\Big) B \qquad (4.1.17)
\end{aligned}
$$

bulunur. $D_N N$ ve $D_N^2 N$ ifadelerinden T ve B nin eşitleri hesaplanırsa,

$$T=\frac{\tau}{\kappa\tau'-\kappa'\tau}D_N^2N-\frac{\tau'}{\kappa\tau'-\kappa'\tau}D_NN+\frac{\tau(\kappa^2+\tau^2)}{\kappa\tau'-\kappa'\tau}N\,,$$

$$B=\frac{\kappa}{\kappa\tau'-\kappa'\tau}D_N^2N-\frac{\kappa'}{\kappa\tau'-\kappa'\tau}D_NN+\frac{\kappa(\kappa^2+\tau^2)}{\kappa\tau'-\kappa'\tau}N$$

olur. Bu ifadeler (4.1.17) bağıntısında yerine yazılırsa D_N^3N nin eşiti

$$\begin{aligned}
D_N^3N &= \Big(\kappa\big(\kappa^2+\tau^2\big)-\kappa''\Big)T-3\Big(\kappa\kappa'+\tau\tau'\Big)N+\Big(\tau''-\big(\kappa^2+\tau^2\big)\tau\Big)B \\
&= \Big(\kappa\big(\kappa^2+\tau^2\big)-\kappa''\Big)\Big(\frac{\tau}{\kappa\tau'-\kappa'\tau}D_N^2N-\frac{\tau'}{\kappa\tau'-\kappa'\tau}D_NN+\frac{\tau\big(\kappa^2+\tau^2\big)}{\kappa\tau'-\kappa'\tau}N\Big) \\
&\quad -3\Big(\kappa\kappa'+\tau\tau'\Big)N \\
&\quad +\Big(\tau''-(\kappa^2+\tau^2)\tau\Big)\Big(\frac{\kappa}{\kappa\tau'-\kappa'\tau}D_N^2N-\frac{\kappa'}{\kappa\tau'-\kappa'\tau}D_NN+\frac{\kappa\big(\kappa^2+\tau^2\big)}{\kappa\tau'-\kappa'\tau}N\Big) \\
&= \Big(\frac{\kappa\tau''-\kappa''\tau}{\kappa\tau'-\kappa'\tau}\Big)D_N^2N+\Big(\frac{\big(\kappa^2+\tau^2\big)\big(\kappa'\tau-\kappa\tau'\big)+\kappa''\tau'-\kappa'\tau''}{\kappa\tau'-\kappa'\tau}\Big)D_NN \\
&\quad \Big(\frac{\big(\kappa^2+\tau^2\big)\big(\kappa\tau''-\kappa''\tau\big)}{\kappa\tau'-\kappa'\tau}-3\big(\kappa\kappa'+\tau\tau'\big)\Big)N
\end{aligned}$$

şeklinde bulunur. Bu ifade N, D_NN, D_N^2N ve D_N^3N nin lineer bileşimi olarak düzenlenir, katsayılarına da sırasıyla n_0, n_1, n_2 ve n_3 denilirse istenilen denklem elde edilir. $\square$

Örnek 4.1.2 Örnek 4.1.1 de verilen β involüt eğrisinin konneksiyona göre diferensiyel denklemini önce β eğrisinin Frenet aparatları cinsinden ardından da α eğrisinin Frenet aparatları türünden yazalım.

$D_{T_\beta}T_\beta$ nin T_β yönünde türevi alınırsa

$$\begin{aligned} D^2_{T_\beta}T_\beta &= D_{T_\beta}\big(D_{T_\beta}T_\beta\big) \\ &= \big(\vartheta\kappa_\beta\big)' D_{T_\beta}T_\beta \frac{1}{\vartheta\kappa_\beta} + \vartheta\kappa_\beta\big(-\vartheta\kappa_\beta T_\beta\big) \end{aligned}$$

veya

$$\begin{aligned} D^2_{T_\beta}T_\beta - \frac{(\vartheta\kappa_\beta)'}{\vartheta\kappa_\beta} D_{T_\beta}T_\beta + (\vartheta\kappa_\beta)^2 T_\beta &= 0, \\ D^2_{T_\beta}T_\beta + T_\beta &= 0 \end{aligned}$$

bulunur. Teorem 4.1.1 kullanılırsa bu denklemin, α eğrisinin Frenet aparatları cinsinden karşılığı:

$$\begin{aligned} D_{T_\beta}T_\beta &= \vartheta\kappa_\beta N_\beta \implies D_N N = -\kappa T + \tau B, \\ D_{T_\beta}N_\beta &= -\vartheta\kappa_\beta T_\beta + 0 \implies D_N\Big(\frac{-1}{\sqrt{2}}T + \frac{1}{\sqrt{2}}B\Big) = -N, \\ D_{T_\beta}B_\beta &= 0 \implies D_N\Big(\frac{1}{\sqrt{2}}T + \frac{1}{\sqrt{2}}B\Big) = 0, \end{aligned}$$

ifadelerinden

$$D_N T = \frac{1}{\sqrt{2}}N, \qquad D_N N = \frac{-1}{\sqrt{2}}T + \frac{1}{\sqrt{2}}B, \qquad D_N B = \frac{-1}{\sqrt{2}}N$$

olur. $D_N N$ nin N yönünde türevi alınırsa

$$
\begin{aligned}
D_N^2 N &= D_N\big(D_N N\big) \\
&= D_N\Big(\frac{-1}{\sqrt{2}}T + \frac{1}{\sqrt{2}}B\Big) \\
&= \frac{-1}{\sqrt{2}}D_N T + \frac{1}{\sqrt{2}}D_N B
\end{aligned}
$$

veya

$$D_N^2 N + N = 0$$

denklemi elde edilir.

4.1.4 İnvolüt Eğrisinin Darboux Vektörüne Göre Denklemleri

Bu kısımda β involüt eğrisinin diferensiyel denklemleri W_β Darboux vektörüne ve C_β birim Darboux vektörüne göre yazılacak, ardından bu denklemlerin esas eğriye bağlı ifadesi, yani α eğrisinin Frenet aparatları cinsinden yazılışları verilecektir.

Teorem 4.1.5 β eğrisi α nın bir involütü ve β eğrisinin Darboux vektörü W_β olsun. β involüt eğrisinin konneksiyona göre diferensiyel denklemi

$$m_{\beta 1} D_{T_\beta}^3 W_\beta + m_{\beta 2} D_{T_\beta}^2 W_\beta + m_{\beta 3} D_{T_\beta} W_\beta + m_{\beta 4} W_\beta = 0 \tag{4.1.18}$$

bağıntısıyla verilir. Burada $m_{\beta 1},\ m_{\beta 2},\ m_{\beta 3}$ ve $m_{\beta 4}$

$$m_{\beta 1} = \vartheta\Big(\kappa_\beta \tau_\beta' - \kappa_\beta' \tau_\beta\Big)^2,$$

$$m_{\beta 2} = \Big(\vartheta \kappa_\beta'' \tau_\beta - \vartheta \kappa_\beta \tau_\beta'' - \big(\vartheta \kappa_\beta \tau_\beta' - \vartheta \kappa_\beta' \tau_\beta\big)'\Big)\Big(\kappa_\beta \tau_\beta' - \kappa_\beta' \tau_\beta\Big),$$

$$
\begin{aligned}
m_{\beta 3} &= \Big(\kappa_\beta'''\tau_\beta - \kappa_\beta\tau_\beta''' + \vartheta^2\big(\kappa_\beta\tau_\beta' - \kappa_\beta'\tau_\beta\big)\big((\kappa_\beta)^2 + (\tau_\beta)^2\big)\Big)\Big(\vartheta\kappa_\beta\tau_\beta' - \vartheta\kappa_\beta'\tau_\beta\Big) \\
&\quad + \Big(\vartheta\kappa_\beta''\tau_\beta - \vartheta\kappa_\beta\tau_\beta'' - \big(\vartheta\kappa_\beta\tau_\beta' - \vartheta\kappa_\beta'\tau_\beta\big)'\Big)\Big(\kappa_\beta''\tau_\beta - \kappa_\beta\tau_\beta''\Big),
\end{aligned}
$$

$$
\begin{aligned}
m_{\beta 4} &= \Big(\kappa_\beta'\tau_\beta''' - \kappa_\beta'''\tau_\beta' - \vartheta^2\big(\kappa_\beta\kappa_\beta' + \tau_\beta\tau_\beta'\big)\big(\kappa_\beta\tau_\beta' - \kappa_\beta'\tau_\beta\big)\Big)\Big(\vartheta\kappa_\beta\tau_\beta' - \vartheta\kappa_\beta'\tau_\beta\Big) \\
&\quad + \Big(\vartheta\kappa_\beta''\tau_\beta - \vartheta\kappa_\beta\tau_\beta'' - \big(\vartheta\kappa_\beta\tau_\beta' - \vartheta\kappa_\beta'\tau_\beta\big)'\big(\kappa_\beta'\tau_\beta'' - \kappa_\beta''\tau_\beta'\big)\Big)
\end{aligned}
$$

şeklinde birer katsayıdır.

İspat. (3.1.5) bağıntısından W_β Darboux vektörü

$$
W_\beta = \tau_\beta T_\beta + \kappa_\beta B_\beta \tag{4.1.19}
$$

dir. Bu vektörün T_β yönünde birinci, ikinci ve üçüncü türevleri sırasıyla,

$$
\begin{aligned}
D_{T_\beta} W_\beta &= D_{T_\beta}\big(\tau_\beta T_\beta + \kappa_\beta B_\beta\big) \\
&= \tau_\beta' T_\beta + \tau_\beta D_{T_\beta} T_\beta + \kappa_\beta' B_\beta + \kappa_\beta D_{T_\beta} B_\beta \\
&= \tau_\beta' T_\beta + \tau_\beta \vartheta\kappa_\beta N_\beta + \kappa_\beta' B_\beta - \kappa_\beta\vartheta\tau_\beta N_\beta \\
&= \tau_\beta' T_\beta + \kappa_\beta' B_\beta,
\end{aligned}
$$

$$
\begin{aligned}
D^2_{T_\beta} W_\beta &= D_{T_\beta}\big(D_{T_\beta} W_\beta\big) \\
&= D_{T_\beta}\big(\tau'_\beta T_\beta + \kappa'_\beta B_\beta\big) \\
&= \tau''_\beta T_\beta + \tau'_\beta D_{T_\beta} T_\beta + \kappa''_\beta B_\beta + \kappa'_\beta D_{T_\beta} B_\beta \\
&= \tau''_\beta T_\beta + \big(\vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta\big)N_\beta + \kappa''_\beta B_\beta, \\
D^3_{T_\beta} W_\beta &= D_{T_\beta}\big(D^2_{T_\beta} W_\beta\big) \\
&= D_{T_\beta}\Big(\tau''_\beta T_\beta + \big(\vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta\big)N_\beta + \kappa''_\beta B_\beta\Big) \\
&= \tau'''_\beta T_\beta + \tau''_\beta \vartheta\kappa_\beta N_\beta + \big(\vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta\big)' N_\beta \\
&\quad + \big(\vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta\big)\big(-\vartheta\kappa_\beta T_\beta + \vartheta\tau_\beta B_\beta\big) + \kappa''_\beta B_\beta - \kappa''_\beta\vartheta\tau_\beta N_\beta \\
&= \Big(\tau'''_\beta - \vartheta^2\kappa_\beta\big(\kappa_\beta\tau'_\beta - \kappa'_\beta\tau_\beta\big)\Big)T_\beta + \Big(\vartheta\kappa_\beta\tau''_\beta - \vartheta\kappa''_\beta\tau_\beta + \big(\vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta\big)'\Big)N_\beta \\
&\quad + \Big(\vartheta^2\tau_\beta\big(\kappa_\beta\tau'_\beta - \kappa'_\beta\tau_\beta\big) + \kappa'''_\beta\Big)B_\beta
\end{aligned}
$$

şeklinde bulunur. W_β ve $D_{T_\beta} W_\beta$ ifadelerinden T_β ve B_β vektörleri çekilirse

$$T_\beta = \frac{\kappa_\beta}{\kappa_\beta\tau'_\beta - \kappa'_\beta\tau_\beta} D_{T_\beta} W_\beta - \frac{\kappa'_\beta}{\kappa_\beta\tau'_\beta - \kappa'_\beta\tau_\beta} W_\beta$$

ve

$$B_\beta = \frac{-\tau_\beta}{\kappa_\beta\tau'_\beta - \kappa'_\beta\tau_\beta} D_{T_\beta} W_\beta + \frac{\tau'_\beta}{\kappa_\beta\tau'_\beta - \kappa'_\beta\tau_\beta} W_\beta$$

olur. Bu ifadeler $D^2_{T_\beta}W_\beta$ türevinde yerine yazılırsa N_β vektörü

$$
\begin{aligned}
N_\beta &= \frac{1}{\vartheta\left(\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta\right)}D^2_{T_\beta}W_\beta+\frac{\kappa''_\beta\tau_\beta-\kappa_\beta\tau''_\beta}{\vartheta\left(\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta\right)^2}D_{T_\beta}W_\beta \\
&\quad +\frac{\kappa'_\beta\tau''_\beta-\kappa''_\beta\tau'_\beta}{\vartheta\left(\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta\right)^2}W_\beta
\end{aligned}
$$

şeklinde bulunur. T_β, B_β ve N_β nin eşitleri $D^3_{T_\beta}W_\beta$ üçüncü türevde yerine yazılırsa

$$
\begin{aligned}
D^3_{T_\beta}W_\beta &= \left(\left(\vartheta\kappa_\beta\tau''_\beta-\vartheta\kappa''_\beta\tau_\beta+\left(\vartheta\kappa_\beta\tau'_\beta-\vartheta\kappa'_\beta\tau_\beta\right)'\right)\left(\frac{1}{\vartheta\kappa_\beta\tau'_\beta-\vartheta\kappa'_\beta\tau_\beta}\right)\right)D^2_{T_\beta}W_\beta \\
&\quad +\left(\left(\kappa_\beta\tau'''_\beta-\kappa'''_\beta\tau_\beta-\vartheta^2\left(\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta\right)\left(\kappa^2_\beta+\tau^2_\beta\right)\right)\left(\frac{1}{\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta}\right)\right. \\
&\quad \left.+\left(\vartheta\kappa_\beta\tau''_\beta-\vartheta\kappa''_\beta\tau_\beta+\left(\vartheta\kappa_\beta\tau'_\beta-\vartheta\kappa'_\beta\tau_\beta\right)'\right)\left(\frac{\kappa''_\beta\tau_\beta-\kappa_\beta\tau''_\beta}{\vartheta\left(\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta\right)^2}\right)\right)D_{T_\beta}W_\beta \\
&\quad +\left(\left(\kappa'''_\beta\tau'_\beta-\kappa'_\beta\tau'''_\beta+\vartheta^2\left(\kappa_\beta\kappa'_\beta+\tau_\beta\tau'_\beta\right)\left(\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta\right)\right)\left(\frac{1}{\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta}\right)\right. \\
&\quad \left.+\left(\vartheta\kappa_\beta\tau''_\beta-\vartheta\kappa''_\beta\tau_\beta+\left(\vartheta\kappa_\beta\tau'_\beta-\vartheta\kappa'_\beta\tau_\beta\right)'\right)\left(\frac{\kappa'_\beta\tau''_\beta-\kappa''_\beta\tau'_\beta}{\vartheta\left(\kappa_\beta\tau'_\beta-\kappa'_\beta\tau_\beta\right)^2}\right)\right)W_\beta
\end{aligned}
$$

şeklinde elde edilir. Bu ifade $D^3_{T_\beta}W_\beta$, $D^2_{T_\beta}W_\beta$, $D_{T_\beta}W_\beta$ ve W_β nin lineer bileşimi olarak düzenlenir ve katsayılarına da $m_{\beta1}$, $m_{\beta2}$, $m_{\beta3}$, $m_{\beta4}$ denilirse, ispat tamamlanır. $\square$

Teorem 4.1.6 β eğrisi α nın bir involütü ve α eğrisinin Darboux vektörü W olsun. Konneksiyona göre β involüt eğrisinin diferensiyel denklemi

$$
\begin{aligned}
&(\frac{m_1}{\lambda\kappa})D_N^3W + \Big(3m_1(\frac{1}{\lambda\kappa})' + \frac{m_2}{\lambda\kappa}\Big)D_N^2W + \Big(3m_1(\frac{1}{\lambda\kappa})'' + 2m_2(\frac{1}{\lambda\kappa})' + \frac{m_3}{\lambda\kappa}\Big)D_NW \\
&+\Big(m_1(\frac{1}{\lambda\kappa})''' + m_2(\frac{1}{\lambda\kappa})'' + m_3(\frac{1}{\lambda\kappa})' + \frac{m_4}{\lambda\kappa}\Big)W \\
&+(\frac{m_1\varphi'}{\lambda\kappa})D_N^3N + \Big(3m_1(\frac{\varphi'}{\lambda\kappa})' + \frac{m_2\varphi'}{\lambda\kappa}\Big)D_N^2N + \Big(3m_1(\frac{\varphi'}{\lambda\kappa})'' + 2m_2(\frac{\varphi'}{\lambda\kappa})' + \frac{m_3\varphi'}{\lambda\kappa}\Big)D_NN \\
&+\Big(m_1(\frac{\varphi'}{\lambda\kappa})''' + m_2(\frac{\varphi'}{\lambda\kappa})'' + m_3(\frac{\varphi'}{\lambda\kappa})' + \frac{m_4\varphi'}{\lambda\kappa}\Big)N = 0 \qquad (4.1.20)
\end{aligned}
$$

bağıntısıyla verilir. Burada $m_1,\ m_2,\ m_3,\ m_4$

$$
\begin{aligned}
m_1 &= \bigg(\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\Big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^{\frac{3}{2}}}\Big)\bigg)^2, \\
m_2 &= \bigg((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}) - \sqrt{\kappa^2+\tau^2}\,(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)})'' \\
&\quad -\Big((\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}})'(\frac{\kappa^2+\tau^2}{\lambda\kappa})\Big)'\bigg)\bigg(\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\Big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\Big)\bigg), \\
m_3 &= \bigg(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'''\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa\big(\kappa^2+\tau^2\big)}\Big) - \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)''' \\
&\quad +(\frac{\kappa\tau'-\kappa'\tau}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}})'\big(\kappa^2+\tau^2\big)\Big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2} + (\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)})^2\Big)\bigg)\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\,\frac{\kappa^2+\tau^2}{\lambda\kappa}
\end{aligned}
$$

$$+\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)-\big(\sqrt{\kappa^2+\tau^2}\,\big)\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''$$

$$-\Big(\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\big)\Big)'\Big)$$

$$\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)-\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,\big)\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''\Big),$$

$$m_4 \quad = \quad \Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'''-\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'''\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'$$

$$-\Big(\sqrt{\kappa^2+\tau^2}\,\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'+\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'\Big)$$

$$\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\big)\Big)\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\big)$$

$$+\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)-\big(\sqrt{\kappa^2+\tau^2}\big)\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''$$

$$-\Big(\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\big)\Big)'$$

$$\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''-\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'\Big)\Big)$$

şeklinde birer katsayıdır.

İspat. (4.1.18) denklemindeki W_β vektörü (3.1.9) bağıntısına göre α eğrisinin Frenet aparatları türünden

$$W_\beta = \frac{1}{\lambda\kappa}W + \frac{\varphi'}{\lambda\kappa}N$$

şeklinde yazılır. Bu vektörün T_β yönündeki birinci, ikinci ve üçüncü türevleri sırasıyla,

$$\begin{aligned}
D_{T_\beta}W_\beta &= D_N\Big(\frac{1}{\lambda\kappa}W + \frac{\varphi'}{\lambda\kappa}N\Big) \\
&= (\frac{1}{\lambda\kappa})'W + \frac{1}{\lambda\kappa}D_N W + (\frac{\varphi'}{\lambda\kappa})'N + \frac{\varphi'}{\lambda\kappa}D_N N \\
&= \frac{1}{\lambda\kappa}D_N W + (\frac{1}{\lambda\kappa})'W + \frac{\varphi'}{\lambda\kappa}D_N N + (\frac{\varphi'}{\lambda\kappa})'N,
\end{aligned}$$

$$\begin{aligned}
D^2_{T_\beta}W_\beta &= D_{T_\beta}\big(D_{T_\beta}W_\beta\big) \\
&= D_N\Big(D_N(\frac{1}{\lambda\kappa}W + \frac{\varphi'}{\lambda\kappa}N)\Big) \\
&= D_N\Big(\frac{1}{\lambda\kappa}D_N W + (\frac{1}{\lambda\kappa})'W + \frac{\varphi'}{\lambda\kappa}D_N N + (\frac{\varphi'}{\lambda\kappa})'N\Big) \\
&= (\frac{1}{\lambda\kappa})'D_N W + (\frac{1}{\lambda\kappa})''W + (\frac{\varphi'}{\lambda\kappa})'D_N N + (\frac{\varphi'}{\lambda\kappa})''N \\
&\quad + \frac{1}{\lambda\kappa}D^2_N W + (\frac{1}{\lambda\kappa})'D_N W + \frac{\varphi'}{\lambda\kappa}D^2_N N + (\frac{\varphi'}{\lambda\kappa})'D_N N
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda\kappa}D_N^2 W + 2\big(\frac{1}{\lambda\kappa}\big)' D_N W + \big(\frac{1}{\lambda\kappa}\big)'' W \\
&\quad + \frac{\varphi'}{\lambda\kappa}D_N^2 N + 2\big(\frac{\varphi'}{\lambda\kappa}\big)' D_N N + \big(\frac{\varphi'}{\lambda\kappa}\big)'' N,
\end{aligned}
$$

$$
\begin{aligned}
D_{T_\beta}^3 W_\beta &= D_{T_\beta}\big(D_{T_\beta}^2 W_\beta\big) \\
&= D_N\Big(D_N^2\big(\frac{1}{\lambda\kappa}W + \frac{\varphi'}{\lambda\kappa}N\big)\Big) \\
&= D_N\Big(\frac{1}{\lambda\kappa}D_N^2 W + 2\big(\frac{1}{\lambda\kappa}\big)' D_N W + \big(\frac{1}{\lambda\kappa}\big)'' W \\
&\quad + \frac{\varphi'}{\lambda\kappa}D_N^2 N + 2\big(\frac{\varphi'}{\lambda\kappa}\big)' D_N N + \big(\frac{\varphi'}{\lambda\kappa}\big)'' N\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \big(\frac{1}{\lambda\kappa}\big)' D_N^2 W + 2\big(\frac{1}{\lambda\kappa}\big)'' D_N W + \big(\frac{1}{\lambda\kappa}\big)''' W \\
&\quad + \big(\frac{\varphi'}{\lambda\kappa}\big)' D_N^2 N + 2\big(\frac{\varphi'}{\lambda\kappa}\big)'' D_N N + \big(\frac{\varphi'}{\lambda\kappa}\big)''' N \\
&\quad + \frac{1}{\lambda\kappa}D_N^3 W + 2\big(\frac{1}{\lambda\kappa}\big)' D_N^2 W + \big(\frac{1}{\lambda\kappa}\big)'' D_N W \\
&\quad + \frac{\varphi'}{\lambda\kappa}D_N^3 N + 2\big(\frac{\varphi'}{\lambda\kappa}\big)' D_N^2 N + \big(\frac{\varphi'}{\lambda\kappa}\big)'' D_N N
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\lambda\kappa}D_N^3 W + 3\big(\frac{1}{\lambda\kappa}\big)' D_N^2 W + 3\big(\frac{1}{\lambda\kappa}\big)'' D_N W + \big(\frac{1}{\lambda\kappa}\big)''' W \\
&\quad + \frac{\varphi'}{\lambda\kappa}D_N^3 N + 3\big(\frac{\varphi'}{\lambda\kappa}\big)' D_N^2 N + 3\big(\frac{\varphi'}{\lambda\kappa}\big)'' D_N N + \big(\frac{\varphi'}{\lambda\kappa}\big)''' N
\end{aligned}
$$

şeklinde bulunur. (4.1.18) denkleminin $m_{\beta 1}$, $m_{\beta 2}$, $m_{\beta 3}$, $m_{\beta 4}$ katsayıları, (3.1.8) bağıntısı kullanılarak tekrar yazılır ve yeni katsayılar sırasıyla $m_1,\ m_2,\ m_3,\ m_4$ ile gösterilirse,

$$
\begin{aligned}
m_1 &= \lambda\kappa\Big(\big(\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)} \frac{\lambda\kappa}{\sqrt{\kappa^2+\tau^2}}\big)' \big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)^2\Big)^2 \\
&= \Big(\big(\frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)' \big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^{\frac{3}{2}}}\big)\Big)^2, \\
m_2 &= \Big(\lambda\kappa\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'' \big(\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big) - \big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)\big(\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''\Big) \\
&\quad -\Big(\lambda\kappa\big(\frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)' \big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\big)\Big)'\Big)\Big(\Big(\frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)' \Big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\Big)\Big) \\
&= \Big(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'' \Big(\frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2}\Big) - \Big(\sqrt{\kappa^2+\tau^2}\Big)\Big(\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)'' \\
&\quad -\Big(\big(\frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)' \big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\big)\Big)'\Big)\Big(\frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)' \Big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\Big), \\
m_3 &= \Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''' \big(\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big) - \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''' \\
&\quad +\big(\lambda\kappa\big)^2\big(\frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)' \big(\frac{\kappa^2+\tau^2}{\big(\lambda\kappa\big)^2}\big)\Big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2} + \big(\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)^2\Big)\Big) \\
&\quad \Big(\big(\frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)' \big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\big)\Big)
\end{aligned}
$$

$$+\Big(\lambda\kappa\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)-\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''\Big)$$

$$-\Big(\lambda\kappa\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\big)\Big)'\Big)$$

$$\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)-\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''\Big)$$

$$= \quad \Big(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'''\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)-\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)'''$$

$$+\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\Big(\kappa^2+\tau^2\Big)\Big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}+\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)^2\Big)\Big)$$

$$\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\,\Big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\Big)+\Big(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)''\,\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)$$

$$-\sqrt{\kappa^2+\tau^2}\,\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)''-\Big(\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\,\big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\big)\Big)'\Big)$$

$$\Big(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)''\,\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)-\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)''\Big)$$

ve

$$
\begin{aligned}
m_4 &= \Bigg(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)''' - \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'''\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)' \\
&\quad -\big(\lambda\kappa\big)^2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)' + \frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'\Big) \\
&\quad \Big(\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\,\big(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\big)\Big)\Bigg)\Big(\lambda\kappa\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\cdot\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\Big) \\
&\quad +\Bigg(\lambda\kappa\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)} - \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)''\Big) \\
&\quad -\Big(\lambda\kappa\big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\big)'\cdot\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\Big)' \\
&\quad \Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'' - \big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'\Big)\Bigg) \\
&= \Bigg(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)''' - \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'''\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)' \\
&\quad -\Big(\sqrt{\kappa^2+\tau^2}\,\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)' + \frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\,\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\big)'\Big) \\
&\quad \Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\Big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\Big)\Big)\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\Big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\Big)
\end{aligned}
$$

$$+\Big(\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)''\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)-\Big(\sqrt{\kappa^2+\tau^2}\Big)\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)''$$

$$-\Big((\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}})'(\frac{\kappa^2+\tau^2}{\lambda\kappa})\Big)'$$

$$\Big((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)})''-(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)})'\Big)\Big)$$

şeklinde bulunur. Bu katsayılarla $D_{T_\beta}W_\beta$, $D^2_{T_\beta}W_\beta$ ve $D^3_{T_\beta}W_\beta$ türevlerinin karşılıkları (4.1.18) bağıntısında yerine yazılırsa, β involüt eğrisinin diferensiyel denklemi α nın Frenet aparatları türünden elde edilir ve bu da ispatı tamamlar. □

Teorem 4.1.7 β eğrisi α nın bir involütü ve β involüt eğrisinin birim Darboux vektörü C_β olsun. β involüt eğrisinin konneksiyona göre diferensiyel denklemi

$$D^3_{T_\beta}C_\beta+r_{\beta 1}D^2_{T_\beta}C_\beta+r_{\beta 2}D_{T_\beta}C_\beta+r_{\beta 3}C_\beta=0 \qquad (4.1.21)$$

bağıntısıyla verilir. Burada $r_{\beta 1}$, $r_{\beta 2}$ ve $r_{\beta 3}$

$$r_{\beta 1} = \Big(\frac{-\phi''_\beta}{\phi'_\beta}\Big)-\frac{\big(\phi'_\beta\,\vartheta\parallel W_\beta\parallel\big)'}{\vartheta\parallel W_\beta\parallel\phi'_\beta},$$

$$r_{\beta 2} = \Big(\vartheta\parallel W_\beta\parallel\Big)^2-\Big(\frac{\phi''_\beta}{\phi'_\beta}\Big)'+\frac{\big(\phi'_\beta\,\vartheta\parallel W_\beta\parallel\big)'}{\vartheta\parallel W_\beta\parallel\big(\phi'_\beta\big)^2}\phi''_\beta+\big(\phi'_\beta\big)^2,$$

$$r_{\beta 3} = \Big(\big(\phi'_\beta\big)^2\Big)'-\frac{\big(\phi'_\beta\,\vartheta\parallel W_\beta\parallel\big)'}{\vartheta\parallel W_\beta\parallel}\phi'_\beta$$

şeklinde birer katsayıdır.

İspat. (3.1.6) bağıntısından C_β birim Darboux vektörü

$$C_\beta \;=\; sin\phi_\beta T_\beta + cos\phi_\beta B_\beta$$

dir. Bu vektörün T_β yönünde türevi

$$\begin{aligned} D_{T_\beta} C_\beta &= D_{T_\beta}\big(sin\phi_\beta T_\beta + cos\phi_\beta B_\beta\big) \\ &= \phi'_\beta\, cos\phi_\beta T_\beta + sin\phi_\beta \vartheta \kappa_\beta N_\beta - \phi'_\beta\, sin\phi_\beta B_\beta - cos\phi_\beta \vartheta \tau_\beta N_\beta \\ &= \phi'_\beta\big(cos\phi_\beta T_\beta - sin\phi_\beta B_\beta\big) \end{aligned}$$

olur. C_β ve $D_{T_\beta} C_\beta$ ifadelerinden, T_β ve B_β vektörleri çekilirse

$$T_\beta = sin\phi_\beta C_\beta + \frac{cos\phi_\beta}{\phi'_\beta} D_{T_\beta} C_\beta \quad \text{ve} \quad B_\beta = cos\phi_\beta C_\beta - \frac{sin\phi_\beta}{\phi'_\beta} D_{T_\beta} C_\beta$$

bulunur. Bu vektörler C_β nin ikinci türevinde yerine yazılırsa,

$$\begin{aligned} D^2_{T_\beta} C_\beta &= D_{T_\beta}\big(D_{T_\beta} C_\beta\big) \\ &= D_{T_\beta}\Big(\phi'_\beta\,\big(cos\phi_\beta T_\beta - sin\phi_\beta B_\beta\big)\Big) \\ &= \phi''_\beta\,\big(cos\phi_\beta T_\beta - sin\phi_\beta B_\beta\big) - \big(\phi'_\beta\big)^2\big(sin\phi_\beta T_\beta + cos\phi_\beta B_\beta\big) \\ &\quad +\phi'_\beta\,\vartheta\big(cos\phi_\beta \kappa_\beta + sin\phi_\beta \tau_\beta\big) N_\beta \end{aligned}$$

$$
\begin{aligned}
&= \phi''_\beta\, cos\phi_\beta\big(sin\phi_\beta C_\beta + \frac{cos\phi_\beta}{\phi'_\beta} D_{T_\beta} C_\beta\big) - \phi''_\beta\, sin\phi_\beta\big(cos\phi_\beta C_\beta - \frac{sin\phi_\beta}{\phi'_\beta} D_{T_\beta} C_\beta\big) \\
&\quad + \phi'_\beta\, \vartheta \parallel W_\beta \parallel N_\beta - \big(\phi'_\beta\big)^2 \\
&= \big(\frac{\phi''_\beta}{\phi'_\beta}\big) D_{T_\beta} C_\beta - (\phi'_\beta)^2 C_\beta + \phi'_\beta\, \vartheta \parallel W_\beta \parallel N_\beta
\end{aligned}
$$

bulunur. Bu eşitlikten N_β vektörü çekilirse

$$
N_\beta = \frac{1}{\vartheta\big(\phi'_\beta\big)^2 \parallel W_\beta \parallel}\Big(\phi'_\beta D^2_{T_\beta} C_\beta - \phi''_\beta D_{T_\beta} C_\beta + (\phi'_\beta)^3 C_\beta\Big)
$$

olur. C_β vektörünün T_β yönünde üçüncü türevi

$$
\begin{aligned}
D^3_{T_\beta} C_\beta &= D_{T_\beta}\Big(\frac{\phi''_\beta}{\phi'_\beta} D_{T_\beta} C_\beta - \big(\phi'_\beta\big)^2 C_\beta + \phi'_\beta\, \vartheta \parallel W_\beta \parallel N_\beta\Big) \\
&= \big(\frac{\phi''_\beta}{\phi'_\beta}\big)' D_{T_\beta} C_\beta + \big(\frac{\phi''_\beta}{\phi'_\beta}\big) D^2_{T_\beta} C_\beta - \big((\phi'_\beta)^2\big)' C_\beta - \big(\phi'_\beta\big)^2 D_{T_\beta} C_\beta \\
&\quad + \vartheta^2\, \phi'_\beta \parallel W_\beta \parallel \big(-\kappa_\beta T_\beta + \tau_\beta B_\beta\big) \\
&\quad + \Big(\vartheta\, \phi'_\beta \parallel W_\beta \parallel\Big)' \frac{1}{\vartheta\big(\phi'_\beta\big)^2 \parallel W_\beta \parallel}\Big(\phi'_\beta D^2_{T_\beta} C_\beta - \phi''_\beta D_{T_\beta} C_\beta + \big(\phi'_\beta\big)^3 C_\beta\Big) \\
&= \Big(\frac{\phi''_\beta}{\phi'_\beta} + \frac{\big(\phi'_\beta\, \vartheta \parallel W_\beta \parallel\big)'}{\vartheta \parallel W_\beta \parallel \phi'_\beta}\Big) D^2_{T_\beta} C_\beta
\end{aligned}
$$

$$+\Big(\big(\frac{\phi''_\beta}{\phi'_\beta}\big)' - \big(\vartheta \parallel W_\beta \parallel\big)^2 - \big(\phi'_\beta\big)^2 - \frac{\big(\phi'_\beta \vartheta \parallel W_\beta \parallel\big)'}{\vartheta \parallel W_\beta \parallel \big(\phi'_\beta\big)^2}\phi''_\beta\Big) D_{T_\beta} C_\beta$$

$$+\Big(\frac{\big(\phi'_\beta \vartheta \parallel W_\beta \parallel\big)'}{\vartheta \parallel W_\beta \parallel}\phi'_\beta - \big((\phi'_\beta)^2\big)'\Big) C_\beta$$

şeklinde bulunur. Bu ifade $D^3_{T_\beta} C_\beta,\ D^2_{T_\beta} C_\beta,\ D_{T_\beta} C_\beta$ ve C_β nin lineer bileşimi olarak düzenlenir ve katsayılarına da sırasıyla $r_{\beta 1},\ r_{\beta 2},\ r_{\beta 3}$ denilirse, istenilen denklem elde edilir. □

Teorem 4.1.8 β eğrisi α nın bir involütü ve α eğrisinin birim Darboux vektörü C olsun. Konneksiyona göre β involüt eğrisinin diferensiyel denklemi

$$\begin{aligned}
& p_1 D^3_N C + \Big(3p'_1 + r_1 p_1\Big) D^2_N C + \Big(3p''_1 + 2r_1 p'_1 + r_2 p_1\Big) D_N C \\
& + \Big(p'''_1 + r_1 p''_1 + r_2 p'_1 + r_3 p_1\Big) C + p_2 D^3_N N + \Big(3p'_2 + r_1 p_2\Big) D^2_N N \\
& + \Big(3p''_2 + 2r_1 p'_2 + r_2 p_2\Big) D_N N + \Big(p'''_2 + r_1 p''_2 + r_2 p'_2 + r_3 p_2\Big) N = 0 \qquad (4.1.22)
\end{aligned}$$

bağıntısıyla verilir. Burada $p_1,\ p_2,\ r_1,\ r_2,\ r_3$

$$p_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}, \qquad p_2 = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}},$$

$$r_1 = -\Bigg(\frac{\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''}{\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'} + \frac{\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'}\Bigg),$$

$$r_2 = (\phi')^2+\kappa^2+\tau^2+\Big((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\Big)^2-\Big(\frac{\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)''}{\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'}\Big)'$$

$$+\frac{\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\Big)^2}\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'',$$

$$r_3 = \Big(\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\Big)^2\Big)'$$

$$-\frac{\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'$$

şeklinde birer katsayıdır.

İspat. (4.1.21) denklemindeki C_β vektörü (3.1.10) bağıntısına göre α eğrisinin Frenet aparatları türünden

$$C_\beta = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}C+\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}N = p_1C+p_2N$$

şeklinde yazılır. Bu vektörün (4.1.1) bağıntısına göre N yönünde birinci, ikinci ve üçüncü türevleri

$$\begin{aligned} D_{T_\beta}C_\beta &= D_N\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}N+\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}C\Big) \\ &= (\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'N+(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'C \\ &\quad +\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D_NN+\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D_NC \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' C \\
&\quad + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' N ,
\end{aligned}
$$

$$
\begin{aligned}
D^2_{T_\beta} C_\beta &= D_{T_\beta}\big(D_{T_\beta} C_\beta\big) \\
&= D_N\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' C \\
&\quad + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' N\Big) \\
&= \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D_N C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'' C \\
&\quad + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D_N N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'' N \\
&\quad + \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D^2_N C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D_N C \\
&\quad + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D^2_N N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D_N N
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^2 C + 2\big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)' D_N C \\
&\quad + \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'' C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^2 N \\
&\quad + 2\big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)' D_N N + \big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'' N \, ,
\end{aligned}
$$

$$
\begin{aligned}
D_{T_\beta}^3 C_\beta &= D_{T_\beta}\big(D_{T_\beta}^2 C_\beta\big) \\
&= D_N\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^2 C + 2\big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)' D_N C \\
&\quad + \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'' C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^2 N \\
&\quad + 2\big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)' D_N N + \big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'' N\Big) \\
&= \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)' D_N^2 C + 2\big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'' D_N C \\
&\quad + \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)''' C + \big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)' D_N^2 N
\end{aligned}
$$

$$+2\left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' D_N N + \left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)''' N$$

$$+\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^3 C + 2\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D_N^2 C$$

$$+\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' D_N C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^3 N$$

$$+2\left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D_N^2 N + \left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' D_N N$$

$$= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^3 C + 3\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D_N^2 C$$

$$+3\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' D_N C + \left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)''' C$$

$$+\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^3 N + 3\left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D_N^2 N$$

$$+3\left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' D_N N + \left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)''' N$$

şeklinde bulunur. Son olarak (4.1.21) denkleminin $r_{\beta 1},\ r_{\beta 2},\ r_{\beta 3}$ katsayıları (3.1.8) bağıntısına göre α eğrisinin eğrilikleri türünden yazılır ve bunlar sırasıyla $r_1,\ r_2,\ r_3$ ile gösterilirse,

$$r_1 = -\Big(\frac{\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)''}{\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'} + \frac{\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'}\Big),$$

$$\begin{aligned} r_2 &= (\phi')^2+\kappa^2+\tau^2+\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\Big)^2-\Big(\frac{\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)''}{\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'}\Big)' \\ &+\frac{\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\Big)^2}\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'', \end{aligned}$$

$$\begin{aligned} r_3 &= \Big(\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\Big)^2\Big)' \\ &-\frac{\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\sqrt{(\phi')^2+\kappa^2+\tau^2}\Big)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' \end{aligned}$$

şeklinde bulunur. Bu katsayılarla $D_{T_\beta}C_\beta$, $D^2_{T_\beta}C_\beta$, $D^3_{T_\beta}C_\beta$ türevlerinin karşılıkları (4.1.21) bağıntısında yerine yazılırsa, β involüt eğrisinin konneksiyona göre diferensiyel denklemi α eğrisinin Frenet aparatları türünden elde edilmiş olur. $\square$

4.1.5 İnvolüt Eğrisinin Normal Darboux Vektörüne Göre Denklemleri

Bu kısımda β involüt eğrisinin diferensiyel denklemleri normal Levi-Civita konneksiyonu kullanılarak ifade edilecektir. Bunun için önce $W_\beta^\perp$ Darboux vektörünün dik tümleyenine göre, sonra $C_\beta^\perp$ birim Darboux vektörünün dik tümleyenine göre yazılacak, ardından bu denklemlerin esas eğriye bağlı ifadesi, yani α eğrisinin Frenet aparatları türünden yazılışları verilecektir.

Teorem 4.1.9 β eğrisi α nın bir involütü olsun. İnvolüt eğrisi β nın normal konneksiyona göre diferensiyel denklemi

$$\begin{aligned}&\Big(\vartheta\big(\kappa_\beta\big)^2\tau_\beta\Big)D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}W^{\perp}_\beta-\kappa_\beta\Big(\big(\vartheta\kappa_\beta\tau_\beta\big)'+\vartheta\kappa'_\beta\tau_\beta\Big)D^{\perp}_{T_\beta}W^{\perp}_\beta\\&+\Big(\kappa'_\beta\big((\vartheta\kappa_\beta\tau_\beta)'+\vartheta\kappa'_\beta\tau_\beta\big)-\vartheta\kappa_\beta\tau_\beta\big(\kappa''_\beta-\vartheta^2\kappa_\beta(\tau_\beta)^2\big)\Big)W^{\perp}_\beta=0\end{aligned} \tag{4.1.23}$$

bağıntısıyla verilir.

İspat. (3.2.3) bağıntısına göre Darboux vektörünün dik tümleyeni $W^{\perp}_\beta=\kappa_\beta B_\beta$ dir. Bu vektörün T_β yönünde normal konneksiyona göre birinci ve ikinci türevi alınırsa,

$$\begin{aligned}D^{\perp}_{T_\beta}W^{\perp}_\beta&=-\vartheta\kappa_\beta\tau_\beta N_\beta+\kappa'_\beta B_\beta,\\D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}W^{\perp}_\beta&=D^{\perp}_{T_\beta}\big(-\vartheta\kappa_\beta\tau_\beta N_\beta+\kappa'_\beta B_\beta\big)\\&=-\Big(\big(\vartheta\kappa_\beta\tau_\beta\big)'+\vartheta\kappa'_\beta\tau_\beta\Big)N_\beta+\Big(\kappa''_\beta-\vartheta^2\kappa_\beta\big(\tau_\beta\big)^2\Big)B_\beta\end{aligned}$$

bulunur. $W^{\perp}_\beta$ ve $D^{\perp}_{T_\beta}W^{\perp}_\beta$ ifadelerinden B_β ve N_β vektörleri çekilirse

$$B_\beta=\big(\frac{1}{\kappa_\beta}\big)W^{\perp}_\beta \quad \text{ve} \quad N_\beta=\big(\frac{-1}{\vartheta\kappa_\beta\tau_\beta}\big)D^{\perp}_{T_\beta}W^{\perp}_\beta+\frac{\kappa'_\beta}{\vartheta\big(\kappa_\beta\big)^2\tau_\beta}W^{\perp}_\beta$$

olur ve bu vektörler $D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}W^{\perp}_\beta$ ikinci türevde yerine yazılırsa, ispat tamamlanır. □

Teorem 4.1.10 β eğrisi α nın bir involütü ve α eğrisinin Darboux vektörü W olsun. β involüt eğrisinin normal konneksiyona göre diferensiyel denklemi

$$\Big(\frac{\kappa\tau'-\kappa'\tau}{(\lambda\kappa)^3}\Big)D_N^{\perp}D_N^{\perp}W+\Big(2(\frac{1}{\lambda\kappa})'\frac{\kappa\tau'-\kappa'\tau}{(\lambda\kappa)^2}-\frac{\sqrt{\kappa^2+\tau^2}}{(\lambda\kappa)^2}\Big((\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa\sqrt{\kappa^2+\tau^2}})'$$
$$+(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)\Big)D_N^{\perp}W$$
$$+\Big(\frac{\kappa\tau'-\kappa'\tau}{(\lambda\kappa)^2}(\frac{1}{\lambda\kappa})''+\frac{\big(\sqrt{\kappa^2+\tau^2}\,\big)'}{(\lambda\kappa)^2}\Big((\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa\sqrt{\kappa^2+\tau^2}})'+(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)$$
$$\Big(\frac{\kappa'\tau-\kappa\tau'}{(\lambda\kappa)^2\sqrt{\kappa^2+\tau^2}}\Big)\Big((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''-\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})^2\Big)\Big)W=0 \tag{4.1.24}$$

bağıntısıyla verilir.

İspat. (4.1.23) denklemindeki $W_\beta^{\perp}$ vektörünün eşiti (3.1.9) bağıntısına göre

$$W_\beta^{\perp}=\frac{1}{\lambda\kappa}W$$

şeklinde yazılır. Bu vektörün (4.1.10) bağıntısına göre N yönünde birinci ve ikinci türevi

$$\begin{aligned}
D_{T_\beta}^{\perp}W_\beta^{\perp} &= D_N^{\perp}\Big(\frac{1}{\lambda\kappa}W\Big)\\
&= (\frac{1}{\lambda\kappa})'W+\frac{1}{\lambda\kappa}D_N^{\perp}W\,,
\end{aligned}$$

$$\begin{aligned}
D_{T_\beta}^{\perp}D_{T_\beta}^{\perp}W_\beta^{\perp} &= D_N^{\perp}D_N^{\perp}\Big(\frac{1}{\lambda\kappa}W\Big)\\
&= (\frac{1}{\lambda\kappa})''W+(\frac{1}{\lambda\kappa})'D_N^{\perp}W+(\frac{1}{\lambda\kappa})'D_N^{\perp}W+\frac{1}{\lambda\kappa}D_N^{\perp}D_N^{\perp}W\\
&= \frac{1}{\lambda\kappa}D_N^{\perp}D_N^{\perp}W+2(\frac{1}{\lambda\kappa})'D_N^{\perp}W+(\frac{1}{\lambda\kappa})''W
\end{aligned}$$

olur. Diğer yandan (4.1.23) denkleminin katsayıları (3.1.8) bağıntısı kullanılarak α eğrisinin eğrilikleri türünden ifade edilirse

$$\vartheta\big(\kappa_\beta\big)^2\tau_\beta = \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)^2\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)}\Big),$$

$$\begin{aligned}\kappa_\beta\Big(\big(\vartheta\kappa_\beta\tau_\beta\big)'+\vartheta\kappa_\beta'\tau_\beta\Big) &= \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\bigg(\Big(\frac{\sqrt{\kappa^2+\tau^2}\big(\kappa\tau'-\kappa'\tau\big)}{\lambda\kappa(\kappa^2+\tau^2)}\Big)' \\ &+\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)}\Big)\bigg),\end{aligned}$$

$$\kappa_\beta'\Big(\big(\vartheta\kappa_\beta\tau_\beta\big)'+\vartheta\kappa_\beta'\tau_\beta\Big)-\vartheta\kappa_\beta\tau_\beta\Big(\kappa_\beta''-\vartheta^2\kappa_\beta\big(\tau_\beta\big)^2\Big)=$$

$$\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\Big(\big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\big)'+\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)}\Big)\cdot$$

$$\frac{\kappa'\tau-\kappa\tau'}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''-\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)$$

ve

$$\kappa_\beta'\Big(\big(\vartheta\kappa_\beta\tau_\beta\big)'+\vartheta\kappa_\beta'\tau_\beta\Big)-\vartheta\kappa_\beta\tau_\beta\Big(\kappa_\beta''-\vartheta^2\kappa_\beta\big(\tau_\beta\big)^2\Big)=$$

$$\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'\bigg(\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\Big)'+\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)\bigg)\cdot$$

$$\Big(\frac{\kappa'\tau-\kappa\tau'}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\Big)\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''-\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)$$

olur. Bu katsayılar (4.1.23) denkleminde yerine yazılırsa

$$\Big(\frac{\kappa\tau' - \kappa'\tau}{(\lambda\kappa)^2}\Big)\Big(\frac{1}{\lambda\kappa}D_N^{\perp}D_N^{\perp}W + 2(\frac{1}{\lambda\kappa})'D_N^{\perp}W + (\frac{1}{\lambda\kappa})''W\Big)$$

$$\Big(\frac{-\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)\Big((\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa\sqrt{\kappa^2+\tau^2}})' + (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2}\Big)\Big((\frac{1}{\lambda\kappa})'W + \frac{1}{\lambda\kappa}D_N^{\perp}W\Big)$$

$$+\Bigg((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\Big((\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa\sqrt{\kappa^2+\tau^2}})' + (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'(\frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2})\Big)$$

$$\Big(\frac{\kappa'\tau - \kappa\tau'}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\Big)'\Big((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'' - (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})(\frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2})^2\Big)\Bigg)\frac{1}{\lambda\kappa}W = 0$$

bulunur. Gerekli düzenlemeler yapılırsa, β involüt eğrisinin normal konneksiyona göre diferensiyel denklemi α eğrisinin Frenet aparatları türünden elde edilmiş olur. □

Teorem 4.1.11 α eğrisinin bir involütü β olsun. β involüt eğrisinin normal konneksiyona göre diferensiyel denklemi

$$s_{\beta 2}D_{T_\beta}^{\perp}D_{T_\beta}^{\perp}C_\beta^{\perp} + s_{\beta 1}D_{T_\beta}^{\perp}C_\beta^{\perp} + s_{\beta 0}C_\beta^{\perp} = 0 \tag{4.1.25}$$

bağıntısıyla verilir. Burada $s_{\beta 2}$, $s_{\beta 1}$ ve $s_{\beta 0}$ katsayıları

$$\begin{aligned}
s_{\beta 2} &= \vartheta\tau_\beta cos^2\phi_\beta, \\
s_{\beta 1} &= cos\phi_\beta\Big(\phi'_\beta sin\phi_\beta\vartheta\tau_\beta - \big(\vartheta\tau_\beta cos\phi_\beta\big)'\Big), \\
s_{\beta 0} &= \phi'_\beta sin\phi_\beta\Big(\phi'_\beta sin\phi_\beta\vartheta\tau_\beta - \big(\vartheta\tau_\beta cos\phi_\beta\big)'\Big) + \vartheta\tau_\beta cos\phi_\beta\Big(\vartheta^2\big(\tau_\beta\big)^2 cos\phi_\beta + \big(\phi'_\beta sin\phi_\beta\big)'\Big).
\end{aligned}$$

İspat. (3.2.3) bağıntısından C_β vektörünün dik tümleyeni $C_\beta^\perp = cos\phi_\beta B_\beta$ dir. Bu vektörün T_β yönünde normal konneksiyona göre birinci ve ikinci türevi

$$D_{T_\beta}^\perp C_\beta^\perp = -\vartheta\tau_\beta cos\phi_\beta N_\beta - \phi_\beta' sin\phi_\beta B_\beta,$$

$$\begin{aligned}
D_{T_\beta}^\perp D_{T_\beta}^\perp C_\beta^\perp &= D_{T_\beta}^\perp\Big(- \vartheta\tau_\beta cos\phi_\beta N_\beta - \big(\phi_\beta\big)' sin\phi_\beta B_\beta\Big) \\
&= \Big(- \vartheta\tau_\beta cos\phi_\beta\Big)' N_\beta - \vartheta\tau_\beta cos\phi_\beta\Big(- \vartheta\kappa_\beta T_\beta + \vartheta\tau_\beta B_\beta\Big) \\
&\quad -\Big(\big(\phi_\beta\big)' sin\phi_\beta\Big)' B_\beta + \Big(\big(\phi_\beta\big)' sin\phi_\beta\Big)\Big(\vartheta\tau_\beta N_\beta\Big) \\
&= \Big(\phi_\beta' sin\phi_\beta\vartheta\tau_\beta - \big(\vartheta\tau_\beta cos\phi_\beta\big)'\Big) N_\beta - \Big(\vartheta^2\big(\tau_\beta\big)^2 cos\phi_\beta + \big(\phi_\beta' sin\phi_\beta\big)'\Big) B_\beta
\end{aligned}$$

olur. $C_\beta^\perp$ ve $D_{T_\beta}^\perp C_\beta^\perp$ ifadelerinden N_β ve B_β vektörleri çekilirse

$$B_\beta = \frac{1}{cos\phi_\beta} C_\beta^\perp \quad \text{ve} \quad N_\beta = \frac{-1}{\vartheta\tau_\beta cos\phi_\beta} D_{T_\beta}^\perp C_\beta^\perp - \frac{\phi_\beta' . sin\phi_\beta}{\vartheta\tau_\beta cos^2\phi_\beta} C_\beta^\perp$$

bulunur. Bu vektörler $C_\beta^\perp$ nin ikinci türevinde yerine yazılarak düzenlenir ve katsayılarına da sırasıyla $s_{\beta 2},\ s_{\beta 1},\ s_{\beta 0}$ denilirse, istenilen denklem elde edilmiş olur. □

Teorem 4.1.12 β eğrisi α nın bir involütü ve α eğrisinin birim Darboux vektörü C olsun. β involüt eğrisinin normal konneksiyona göre diferensiyel denklemi

$$p_1\varphi_2 D_N^{\perp} D_N^{\perp} C + \Big(2p_1'\varphi_2 + p_1\varphi_1\Big) D_N^{\perp} C + \Big(p_1''\varphi_2 + p_1'\varphi_1 + p_1\varphi_0\Big) C = 0 \qquad (4.1.26)$$

bağıntısıyla verilir. Burada $\varphi_0,\ \varphi_1,\ \varphi_2$ ve p_1

$$\begin{aligned}
\varphi_0 \;=\; & \Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big) \\
& \Bigg(\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\Big(\frac{\phi'(\kappa\tau'-\kappa'\tau)}{(\kappa^2+\tau^2)\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big) \\
& -\Big(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\Big)'\Bigg) \\
& +\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\Bigg(\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)}\Big)^2\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} \\
& +\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\Bigg),
\end{aligned}$$

$$\begin{aligned}
\varphi_1 \;=\; & \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Bigg(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\frac{\phi'\big(\kappa\tau'-\kappa'\tau\big)}{(\kappa^2+\tau^2)\sqrt{(\phi')^2+\kappa^2+\tau^2}} \\
& -\big(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\big)'\Bigg),
\end{aligned}$$

$$\varphi_2 \;=\; \frac{\kappa\tau'-\kappa'\tau}{(\phi')^2+\kappa^2+\tau^2} \quad \text{ve} \quad p_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}$$

şeklinde birer katsayıdır.

İspat. (4.1.25) denklemindeki $C_\beta^\perp$ vektörü (3.1.10) bağıntısından

$$C_\beta^\perp = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} C = p_1 C$$

şeklinde yazılır. Bu vektörün (4.1.10) bağıntısına göre N yönünde birinci ve ikinci türevi

$$\begin{aligned}
D_{T_\beta}^\perp C_\beta^\perp &= D_N^\perp \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} C \Big) \\
&= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^\perp C + \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} \big)' C\,, \\
D_{T_\beta}^\perp D_{T_\beta}^\perp C_\beta^\perp &= D_N^\perp \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^\perp C + \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} \big)' C \Big) \\
&= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} D_N^\perp D_N^\perp C + 2 \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} \big)' D_N^\perp C \\
&\quad + \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} \big)'' C
\end{aligned}$$

olur. Diğer yandan (4.1.25) denkleminin $s_{\beta 2}$, $s_{\beta 1}$, $s_{\beta 0}$ katsayıları (3.1.8) bağıntısına göre α eğrisinin eğrilikleri cinsinden yazılır ve bunlar sırasıyla φ_2, φ_1, φ_0 ile gösterilirse,

$$\begin{aligned}
\varphi_2 &= \lambda\kappa \frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)} \big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} \big)^2 \\
&= \frac{\kappa\tau' - \kappa'\tau}{(\phi')^2+\kappa^2+\tau^2}\,,
\end{aligned}$$

$$\begin{aligned}
\varphi_1 &= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\frac{\phi'\big(\kappa\tau'-\kappa'\tau\big)}{(\kappa^2+\tau^2)\sqrt{(\phi')^2+\kappa^2+\tau^2}} \\
&\quad -\big(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\big)'\Big),
\end{aligned}$$

$$\begin{aligned}
\varphi_0 &= \Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big) \\
&\quad \bigg(\Big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\Big(\frac{\phi'(\kappa\tau'-\kappa'\tau)}{(\kappa^2+\tau^2)\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big) \\
&\quad -\Big(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\Big)'\bigg) \\
&\quad +\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\bigg(\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)}\Big)^2\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}} \\
&\quad +\Big(\big(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\big)'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\bigg)
\end{aligned}$$

şeklinde bulunur. Bu katsayılarla $C_\beta^\perp$, $D_{T_\beta}^\perp C_\beta^\perp$, $D_{T_\beta}^\perp D_{T_\beta}^\perp C_\beta^\perp$ türevlerinin eşitleri (4.1.25) bağıntısında yerine yazılırsa, involüt eğrisinin diferensiyel denklemi α eğrisinin Frenet aparatları cinsinden elde edilmiş olur. $\square$

4.1.6 İnvolüt Eğrisinin Darboux Vektörüne Göre Harmonikliği

(3.2.2) ve (3.2.4) bağıntılarında H ortalama eğrilik vektörü yerine, W Darboux vektörü yazılırsa, biharmonik eğrilerin sınıflandırılması eğrinin Darboux vektörüne göre yeniden yorumlanabilir.

Teorem 4.1.13 β eğrisi α nın bir involütü olsun. Konneksiyona göre β involüt eğrisinin harmoniklik koşulları, W_β Darboux vektörü türünden aşağıdaki önermelerle verilir:
1. β involüt eğrisi Darboux vektörüne göre biharmoniktir ancak ve ancak

$$\tau''_\beta = 0, \quad \vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta = 0\,, \quad \kappa''_\beta = 0,$$

2. β involüt eğrisi Darboux vektörüne göre 1.tipten harmoniktir ancak ve ancak

$$\tau''_\beta = -\lambda_0\tau_\beta, \quad \vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta = 0\,,\ \kappa''_\beta = -\lambda_0\kappa_\beta\,, \quad \lambda_0 \in \mathbb{R}.$$

İspat. Lemma 4.0.9 dan Darboux vektörünün Laplace görüntüsü

$$\begin{aligned}\Delta W_\beta &= -D^2_{T_\beta}W_\beta \\ &= -\tau''_\beta T_\beta - (\vartheta\kappa_\beta\tau'_\beta - \vartheta\kappa'_\beta\tau_\beta)N_\beta - \kappa''_\beta B_\beta\end{aligned}$$

bulunur. $\Delta W_\beta = 0$ ise birinci önerme, $\Delta W_\beta = \lambda_0 W_\beta$ ise ikinci önerme sağlanır. □

Teorem 4.1.14 β eğrisi α nın bir involütü olsun. Konneksiyona göre β involüt eğrisinin harmoniklik koşulları, C_β birim Darboux vektörü türünden aşağıdaki önermelerle verilir:
1. β involüt eğrisi birim Darboux vektörüne göre biharmoniktir ancak ve ancak

$$\phi''_\beta cos\phi_\beta - (\phi'_\beta)^2 sin\phi_\beta = 0, \quad \phi''_\beta sin\phi_\beta + (\phi'_\beta)^2 cos\phi_\beta = 0\,, \quad \phi'_\beta\vartheta \parallel W_\beta \parallel = 0,$$

2. β involüt eğrisi birim Darboux vektörüne göre 1.tipten harmoniktir ancak ve ancak

$$\phi''_\beta cos\phi_\beta - (\phi'_\beta)^2 sin\phi_\beta = -\lambda_0 sin\phi_\beta, \quad \phi''_\beta sin\phi_\beta + (\phi'_\beta)^2 cos\phi_\beta = 0$$

ve $\quad \phi'_\beta \vartheta \parallel W_\beta \parallel = -\lambda_0 cos\phi_\beta\,, \quad \lambda_0 \in \mathbb{R}$ dir.

İspat. Lemma 4.0.10 bağıntısından birim Darboux vektörünün Laplace görüntüsü

$$\begin{aligned}\Delta C_\beta &= -D^2_{T_\beta} C_\beta \\ &= \big((\phi'_\beta)^2 sin\phi_\beta - \phi''_\beta cos\phi_\beta\big)T_\beta + \big(\phi''_\beta sin\phi_\beta + (\phi'_\beta)^2 cos\phi_\beta\big)N_\beta - \big(\phi'_\beta \vartheta \parallel W_\beta \parallel\big)B_\beta\end{aligned}$$

bulunur. Burada $\Delta C_\beta = 0$ ise birinci önerme, $\Delta C_\beta = \lambda_0 C_\beta$ ise ikinci önerme sağlanır. □

4.1.7 İnvolüt Eğrisinin Ortalama Eğrilik Vektörüne Göre Denklemleri

Bu kısımda β involüt eğrisinin diferensiyel denklemleri önce H_β ortalama eğrilik vektörüne göre yazılacak, ardından bu denklemlerin esas eğriye bağlı ifadesi, yani α eğrisinin Frenet aparatları cinsinden yazılışları verilecektir.

Teorem 4.1.15 β eğrisi α nın bir involütü ve β nın ortalama eğrilik vektörü H_β olsun. Konneksiyona göre β eğrisinin diferensiyel denklemi

$$h_{\beta 1} D^3_{T_\beta} H_\beta + h_{\beta 2} D^2_{T_\beta} H_\beta + h_{\beta 3} D_{T_\beta} H_\beta + h_{\beta 4} H_\beta = 0 \tag{4.1.27}$$

bağıntısıyla verilir. Burada $h_{\beta 1},\ h_{\beta 2},\ h_{\beta 3}$ ve $h_{\beta 4}$

$$\begin{aligned}
h_{\beta 1} &= -\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\Big(\vartheta^2 \kappa_\beta \tau_\beta\Big)^2, \\
h_{\beta 2} &= 3\vartheta^2 \kappa_\beta \tau_\beta \Big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)' - \big(\vartheta\kappa_\beta\big)^2 \Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)', \\
h_{\beta 3} &= \Big(\vartheta^4\kappa_\beta^2\big(\kappa_\beta^2+\tau_\beta^2\big) - \vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - 3\Big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)'\Big)\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big) \\
&\quad + \Big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big) + \big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'\big)'\Big)\Big(3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big),
\end{aligned}$$

$$
\begin{aligned}
h_{\beta 4} \;\; = \;\; & \Big(\Big(\vartheta^4\kappa_\beta^2(\kappa_\beta^2+\tau_\beta^2) - \vartheta\kappa_\beta(\vartheta\kappa_\beta)'' - 3\big(\vartheta\kappa_\beta(\vartheta\kappa_\beta)'\big)'\Big)\Big) \\
& \Big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big) - \frac{(\vartheta\kappa_\beta)'}{\vartheta\kappa_\beta}\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)\Big) \\
& +\Big(\Big(\big(\vartheta\kappa_\beta\big)'' - \vartheta^3\kappa_\beta\big(\kappa_\beta^2+\tau_\beta^2\big)\Big)' - 3\big(\vartheta\kappa_\beta\big)^2\big(\vartheta\kappa_\beta\big)' - 2\big(\vartheta\tau_\beta\big)^2\big(\vartheta\kappa_\beta\big)' \\
-\vartheta^2\kappa_\beta\tau_\beta\big(\vartheta\tau_\beta\big)'\Big)&\Big(\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\vartheta^3\kappa_\beta\tau_\beta^2\Big) + \Big(\Big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big) \\
+\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' &+ \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)'\Big)\Big)\Big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta^2\big(\kappa_\beta^2+\tau_\beta^2\big) - 3\Big(\big(\vartheta\kappa_\beta\big)'\Big)^2\Big)
\end{aligned}
$$

şeklinde birer katsayıdır.

İspat. (3.2.1) bağıntısından $H_\beta = \vartheta\kappa_\beta N_\beta$ yazılır. Bu vektörün T_β yönünde birinci, ikinci ve üçüncü türevi

$$
\begin{aligned}
D_{T_\beta}H_\beta &= D_{T_\beta}\Big(\vartheta\kappa_\beta N_\beta\Big) \\
&= -\big(\vartheta^2\kappa_\beta^2\big)T_\beta + \big(\vartheta\kappa_\beta\big)'N_\beta + \big(\vartheta^2\kappa_\beta\tau_\beta\big)B_\beta\,,
\end{aligned}
$$

$$
\begin{aligned}
D_{T_\beta}^2H_\beta &= D_{T_\beta}D_{T_\beta}\Big(\vartheta\kappa_\beta N_\beta\Big) \\
&= D_{T_\beta}\Big(-\vartheta^2\kappa_\beta^2T_\beta + \big(\vartheta\kappa_\beta\big)'N_\beta + \vartheta^2\kappa_\beta\tau_\beta B_\beta\Big) \\
&= -\big(\vartheta^2\kappa_\beta^2\big)'T_\beta + \big((\vartheta\kappa_\beta)'\big)'N_\beta + \big(\vartheta^2\kappa_\beta\tau_\beta\big)'B_\beta \\
&\quad -\vartheta^2\kappa_\beta^2D_{T_\beta}T_\beta + \big(\vartheta\kappa_\beta\big)'D_{T_\beta}N_\beta + \vartheta^2\kappa_\beta\tau_\beta D_{T_\beta}B_\beta
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(-3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)T_\beta + \Big(\big(\vartheta\kappa_\beta\big)'' - \big(\vartheta\kappa_\beta\big)^3 - \vartheta^3\kappa_\beta\tau_\beta^2\Big)N_\beta \\
&\quad +\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)B_\beta,
\end{aligned}
$$

$$
\begin{aligned}
D^3_{T_\beta}H_\beta &= D_{T_\beta}\Bigg(\Big(-3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)T_\beta + \Big(\big(\vartheta\kappa_\beta\big)'' - \big(\vartheta\kappa_\beta\big)^3 - \vartheta^3\kappa_\beta\tau_\beta^2\Big)N_\beta \\
&\quad +\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)B_\beta\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(-3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)'T_\beta + \Big(-3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)D_{T_\beta}T_\beta \\
&\quad +\Big(\big(\vartheta\kappa_\beta\big)'' - \big(\vartheta\kappa_\beta\big)^3 - \vartheta^3\kappa_\beta\tau_\beta^2\Big)'N_\beta + \Big(\big(\vartheta\kappa_\beta\big)'' - \big(\vartheta\kappa_\beta\big)^3 - \vartheta^3\kappa_\beta\tau_\beta^2\Big)D_{T_\beta}N_\beta \\
&\quad +\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)'B_\beta + \Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)D_{T_\beta}B_\beta
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\vartheta^4\kappa_\beta^2\big(\kappa_\beta^2+\tau_\beta^2\big) - \vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - 3\Big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)'\Big)T_\beta \\
&\quad +\Bigg(\Big(\big(\vartheta\kappa_\beta\big)'' - \vartheta^3\kappa_\beta\big(\kappa_\beta^2+\tau_\beta^2\big)\Big)' - 3\big(\vartheta\kappa_\beta\big)^2\big(\vartheta\kappa_\beta\big)' \\
&\quad -2\big(\vartheta\tau_\beta\big)^2\big(\vartheta\kappa_\beta\big)' - \vartheta^2\kappa_\beta\tau_\beta\big(\vartheta\tau_\beta\big)'\Bigg)N_\beta \\
&\quad +\Bigg(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big) + \Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)'\Bigg)B_\beta
\end{aligned}
$$

şeklinde bulunur. $D_{T_\beta}H_\beta$ ve $D^2_{T_\beta}H_\beta$ ifadelerinden T_β ve B_β vektörleri çekilirse

$$T_\beta = \frac{\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big) - \frac{\left(\vartheta\kappa_\beta\right)'}{\vartheta\kappa_\beta}\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)}{\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\big(\vartheta^2\kappa_\beta\tau_\beta\big)^2} H_\beta$$

$$+\frac{2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'}{\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\big(\vartheta^2\kappa_\beta\tau_\beta\big)^2} D_{T_\beta}H_\beta - \frac{1}{\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\big(\vartheta^2\kappa_\beta\tau_\beta\big)} D^2_{T_\beta}H_\beta\,,$$

$$B_\beta = \frac{\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta^2\big(\kappa_\beta^2+\tau_\beta^2\big) - 3\big((\vartheta\kappa_\beta)'\big)^2}{\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\big(\vartheta^2\kappa_\beta\tau_\beta\big)^2} H_\beta + \frac{3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'}{\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\big(\vartheta^2\kappa_\beta\tau_\beta\big)^2} D_{T_\beta}H_\beta$$

$$-\frac{1}{\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\big(\vartheta\tau_\beta\big)^2} D^2_{T_\beta}H_\beta$$

olur. Bu vektörler $D^3_{T_\beta}H_\beta$ ifadesinde yerine yazılırsa

$$\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\Big(\vartheta^2\kappa_\beta\tau_\beta\Big)^2 D^3_{T_\beta}H_\beta = \Bigg(\Big(\vartheta^4\kappa_\beta^2\big(\kappa_\beta^2+\tau_\beta^2\big) - \vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - 3\big(\vartheta\kappa_\beta(\vartheta\kappa_\beta)'\big)'\Big)$$

$$\Big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big) - \frac{\big(\vartheta\kappa_\beta\big)'}{\vartheta\kappa_\beta}\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)\Big)$$

$$+\bigg(\Big(\Big(\big(\vartheta\kappa_\beta\big)'' - \vartheta^3\kappa_\beta\big(\kappa_\beta^2+\tau_\beta^2\big)\Big)' - 3\big(\vartheta\kappa_\beta\big)^2\big(\vartheta\kappa_\beta\big)' - 2\big(\vartheta\tau_\beta\big)^2\big(\vartheta\kappa_\beta\big)' - \vartheta^2\kappa_\beta\tau_\beta\big(\vartheta\tau_\beta\big)'\Big)$$

$$\Big(\big(\frac{\kappa_\beta}{\tau_\beta}\big)'\vartheta^3\kappa_\beta\tau_\beta^2\Big) + \bigg(\Big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big) + \Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)'\Big)\bigg)$$

$$\Big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta^2\big(\kappa_\beta^2+\tau_\beta^2\big) - 3\big((\vartheta\kappa_\beta)'\big)^2\Big)\Bigg)H_\beta$$

$$+\Bigg(\Big(\vartheta^4\kappa_\beta^2\big(\kappa_\beta^2+\tau_\beta^2\big)-\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)''-3\big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\big)'\Big)\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'+\vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)$$

$$+\Bigg(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)''-\vartheta^4\kappa_\beta\tau_\beta\big(\kappa_\beta^2+\tau_\beta^2\big)$$

$$+\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'+\vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)'\Bigg)\Big(3\vartheta\kappa_\beta(\vartheta\kappa_\beta)'\Big)\Bigg)D_{T_\beta}H_\beta$$

$$+\Bigg(3\vartheta^2\kappa_\beta\tau_\beta\Big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big)'-\big(\vartheta\kappa_\beta\big)^2\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'+\vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)'\Bigg)D_{T_\beta}^2H_\beta$$

elde edilir. Bu ifade $D_{T_\beta}^3H_\beta,\ D_{T_\beta}^2H_\beta,\ D_{T_\beta}H_\beta$ ve H_β nin lineer bileşimi olarak düzenlenir ve katsayılarına da sırasıyla, $h_{\beta1},\ h_{\beta2},\ h_{\beta3}$ ve $h_{\beta4}$ denilirse, ispat tamamlanır. □

Teorem 4.1.16 β eğrisi α nın bir involütü ve α eğrisinin asli normali N olsun. Konneksiyona göre β eğrisinin diferensiyel denklemi

$$h_1D_N^4N+h_2D_N^3N+h_3D_N^2N+h_4D_NN=0$$

bağıntısıyla verilir. Burada $h_1,\ h_2,\ h_3,\ h_4$ katsayıları

$$h_1 = \Big(\frac{\big(\kappa^2+\tau^2\big)^{3/2}}{\kappa'\tau-\kappa\tau'}\Big)'\Big(\frac{\big(\kappa\tau'-\kappa'\tau\big)^2}{\kappa^2+\tau^2}\Big),$$

$$h_2 = 3\Big(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big)\Big(\kappa\kappa'+\tau\tau'\Big)'-\Big(\kappa^2+\tau^2\Big)\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big(\sqrt{\kappa^2+\tau^2}\,\big)'$$

$$+\sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)',$$

$$
\begin{aligned}
h_3 \;=\;& \Bigg(\Big(\kappa^2+\tau^2\Big)\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-\sqrt{\kappa^2+\tau^2}\big(\sqrt{\kappa^2+\tau^2}\,\big)''\\
&-3\Big(\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'\Big)'\Bigg)\Bigg(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'+\sqrt{\kappa^2+\tau^2}\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)'\Bigg)\\
&+\Bigg(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}''-\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)\\
&+\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'+\sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)'\Bigg)\Bigg(3\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
h_4 \;=\;& \Bigg(\Big(\kappa^2+\tau^2\Big)^2+\frac{\Big(\kappa\tau'-\kappa'\tau\Big)^2}{\kappa^2+\tau^2}-\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}''-3\Big(\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'\Big)'\Bigg)\\
&\Bigg(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}''-\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)\\
&-\Big(\frac{\kappa\kappa'+\tau\tau'}{\kappa^2+\tau^2}\Big)\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'+\sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)\Bigg)\\
&+\Bigg(\Big(\sqrt{\kappa^2+\tau^2}''-\sqrt{\kappa^2+\tau^2}\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)\Big)'-3\Big(\kappa^2+\tau^2\Big)\sqrt{\kappa^2+\tau^2}'\\
&-2\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)^2\sqrt{\kappa^2+\tau^2}'-\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)'\Bigg)\\
&\Bigg(\Big(\frac{\big(\kappa^2+\tau^2\big)^{3/2}}{\kappa\tau'-\kappa'\tau}\Big)'\Big(\frac{\Big(\kappa\tau'-\kappa'\tau\Big)^2}{\Big(\kappa^2+\tau^2\Big)^{3/2}}\Big)\Bigg)\\
&+\Bigg(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}''-\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)\\
&+\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'+\sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)'\Bigg)\\
&\Bigg(\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}''-\Big(\kappa^2+\tau^2\Big)^2-\frac{\Big(\kappa\tau'-\kappa'\tau\Big)^2}{\kappa^2+\tau^2}-3\Big(\sqrt{\kappa^2+\tau^2}'\Big)^2\Bigg).
\end{aligned}
$$

İspat. (4.1.27) denklemindeki H_β ifadesi (3.1.7) ve (3.1.8) bağıntılarına göre α eğrisinin Frenet aparatları türünden yazılırsa,

$$\begin{aligned} H_\beta &= D_{T_\beta}T_\beta \\ &= \vartheta\kappa_\beta N_\beta \\ &= \lambda\kappa\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big(\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T+\frac{\tau}{\sqrt{\kappa^2+\tau^2}}B\Big) \\ &= D_N N \end{aligned}$$

olur. Bu vektörün N yönünde birinci, ikinci ve üçüncü türevi

$$\begin{aligned} D_{T_\beta}H_\beta &= D_N\big(H_\beta\big) \\ &= D_N\big(D_N N\big) \\ &= D_N^2 N, \\ D_{T_\beta}^2 H_\beta &= D_N^2\big(H_\beta\big) \\ &= D_N^2\big(D_N N\big) \\ &= D_N^3 N, \\ D_{T_\beta}^3 H_\beta &= D_N^3\big(H_\beta\big) \\ &= D_N^3\big(D_N N\big) \\ &= D_N^4 N \end{aligned}$$

şeklinde bulunur. Son olarak (4.1.27) denkleminin $h_{\beta 1},\ h_{\beta 2},\ h_{\beta 3},\ h_{\beta 4}$ katsayıları (3.1.8) bağıntısına göre α eğrisinin eğrilikleri türünden yazılır ve bunlar sırasıyla, $h_1,\ h_2,\ h_3,\ h_4$ ile gösterilirse,

$$h_1 = -\Big(\frac{\big(\kappa^2+\tau^2\big)^{3/2}}{\kappa\tau'-\kappa'\tau}\Big)'\frac{\big(\kappa\tau'-\kappa'\tau\big)^2}{\kappa^2+\tau^2},$$

$$\begin{aligned} h_2 = {} & 3\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\big(\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'\big)' - \big(\kappa^2+\tau^2\big)\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}' \\ & +\sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)', \end{aligned}$$

$$\begin{aligned} h_3 = {} & \Big(\big(\kappa^2+\tau^2\big)\big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\big) - \sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'' \\ & -3\big(\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'\big)'\Big)\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}' + \sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big) \\ & +\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'' - \frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big) \\ & +\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}' + \sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)'\Big)\Big(3\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'\Big), \end{aligned}$$

$$\begin{aligned} h_4 = {} & \Big(\big(\kappa^2+\tau^2\big)^2 + \frac{\big(\kappa\tau'-\kappa'\tau\big)^2}{\kappa^2+\tau^2} - \sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'' - 3\big(\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'\big)'\Big) \\ & \Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'' - \frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\big) \end{aligned}$$

$$-\frac{\sqrt{\kappa^2+\tau^2}'}{\sqrt{\kappa^2+\tau^2}}\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'+\sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)\Big)$$

$$+\bigg(\Big(\sqrt{\kappa^2+\tau^2}''-\sqrt{\kappa^2+\tau^2}\big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\big)\Big)'-3\big(\kappa^2+\tau^2\big)\sqrt{\kappa^2+\tau^2}'$$

$$-2\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\sqrt{\kappa^2+\tau^2}'-\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\bigg)\bigg(\Big(\frac{(\kappa^2+\tau^2)^{3/2}}{\kappa\tau'-\kappa'\tau}\Big)'\frac{\big(\kappa\tau'-\kappa'\tau\big)^2}{\big(\kappa^2+\tau^2\big)^{3/2}}\bigg)$$

$$+\bigg(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}''-\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)$$

$$+\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}'+\sqrt{\kappa^2+\tau^2}\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)'\bigg)\bigg(\sqrt{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}''$$

$$-\big(\kappa^2+\tau^2\big)^2-\frac{\big(\kappa\tau'-\kappa'\tau\big)^2}{\kappa^2+\tau^2}-3\big(\sqrt{\kappa^2+\tau^2}'\big)^2\bigg)$$

şeklinde bulunur. Bu katsayılarla $D_{T_\beta}H_\beta,\ D^2_{T_\beta}H_\beta,\ D^3_{T_\beta}H_\beta$ türevlerinin eşitleri (4.1.27) bağıntısında yerine yazılırsa, β involüt eğrisinin deiferensiyel denklemi α eğrisinin Frenet aparatları türünden elde edilmiş olur. □

Teorem 4.1.17 α eğrisinin bir involütü β ve involüt eğrisi β nın ortalama eğrilik vektörü H_β olsun. Normal konneksiyona göre β eğrisinin diferensiyel denklemi

$$\eta_{\beta 1}D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}H^{\perp}_\beta+\eta_{\beta 2}D^{\perp}_{T_\beta}H^{\perp}_\beta+\eta_{\beta 3}H^{\perp}_\beta=0 \qquad (4.1.28)$$

bağıntısıyla verilir. Burada $\eta_{\beta 1},\ \eta_{\beta 2},\ \eta_{\beta 3}$

$$\eta_{\beta 1} = \big(\vartheta\kappa_\beta\big)^2\vartheta\tau_\beta\,,$$

$$\eta_{\beta 2} = -2\vartheta^2\kappa_\beta\tau_\beta(\vartheta\kappa_\beta)'+\big(\vartheta\kappa_\beta\big)^2\big(\vartheta\tau_\beta\big)'\,,$$

$$\eta_{\beta 3} = 2\Big(\big(\vartheta\kappa_\beta\big)'\Big)^2\vartheta\tau_\beta+\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\big(\vartheta\tau_\beta\big)'+\big(\vartheta\kappa_\beta\big)^2\big(\vartheta\tau_\beta\big)^3-\vartheta^2\kappa_\beta\tau_\beta\big(\vartheta\kappa_\beta\big)''$$

şeklinde birer katsayıdır.

İspat. (3.2.3) bağıntısına göre H_β nin dik tümleyeni

$$H_\beta^\perp = \vartheta\kappa_\beta N_\beta$$

olur. Bu vektörün T_β yönünde normal konneksiyona göre birinci ve ikinci türevi

$$\begin{aligned}
D_{T_\beta}^\perp H_\beta^\perp &= D_{T_\beta}^\perp\big(\vartheta\kappa_\beta N_\beta\big)\\
&= \big(\vartheta\kappa_\beta\big)' N_\beta + \big(\vartheta^2\kappa_\beta\tau_\beta\big)B_\beta\,,\\
D_{T_\beta}^\perp D_{T_\beta}^\perp H_\beta^\perp &= D_{T_\beta}^\perp\Big(\big(\vartheta\kappa_\beta\big)' N_\beta + \big(\vartheta^2\kappa_\beta\tau_\beta\big)B_\beta\Big)\\
&= \Big(\big(\vartheta\kappa_\beta\big)'' - \vartheta^3\kappa_\beta\tau_\beta^2\Big)N_\beta + \Big(2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)B_\beta
\end{aligned}$$

olur. $H_\beta^\perp$ ve $D_{T_\beta}^\perp H_\beta^\perp$ ifadelerinden N_β ve B_β vektörleri hesaplanırsa

$$N_\beta = \frac{1}{\vartheta\kappa_\beta}H_\beta^\perp \quad \text{ve} \quad B_\beta = \frac{1}{\vartheta^2\kappa_\beta\tau_\beta}D_{T_\beta}^\perp H_\beta^\perp - \frac{(\vartheta\kappa_\beta)'}{\vartheta^3\kappa_\beta^2\tau_\beta}H_\beta^\perp$$

bulunur. Bu ifadeler $D_{T_\beta}^\perp D_{T_\beta}^\perp H_\beta^\perp$ türevinde yerine yazılırsa β involüt eğrisinin normal konneksiyona göre diferensiyel denklemi elde edilmiş olur. □

4.1.8 İnvolüt Eğrisinin Ortalama Eğrilik Vektörüne Göre Harmonikliği

Teorem 4.1.18 β eğrisi α nın bir involütü olsun. Konneksiyona göre β involüt eğrisinin harmoniklik koşulları aşağıdaki önermelerle verilir:

1. β involüt eğrisinin biharmonik olması için gerek ve yeter koşul

$$\big(\vartheta\kappa_\beta\big)'\vartheta\kappa_\beta = 0, \quad \big(\vartheta\kappa_\beta\big)^3 + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)^2 - \big(\vartheta\kappa_\beta\big)'' = 0 \;\text{ ve }\; -2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta - \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)' = 0.$$

2. β involüt eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$\big(\vartheta\kappa_\beta\big)'\vartheta\kappa_\beta = 0,\ \big(\vartheta\kappa_\beta\big)^3 + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)^2 - \big(\vartheta\kappa_\beta\big)'' = \lambda_0\vartheta\kappa_\beta\,,\ -2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta - \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)' = 0.$$

İspat. $H_\beta = \vartheta\kappa_\beta N_\beta$ vektörünün (3.2.2) bağıntısına göre Laplace görüntüsü

$$\begin{aligned}
\Delta H_\beta &= -D^2_{T_\beta}\big(\vartheta\kappa_\beta N_\beta\big) \\
&= -D_{T_\beta}\Big(D_{T_\beta}\big(\vartheta\kappa_\beta N_\beta\big)\Big) \\
&= -D_{T_\beta}\Big(\big(\vartheta\kappa_\beta\big)'N_\beta + \big(\vartheta\kappa_\beta\big)D_{T_\beta}N_\beta\Big) \\
&= -\Big(\big(\vartheta\kappa_\beta\big)''N_\beta + 2\big(\vartheta\kappa_\beta\big)'\big(-\vartheta\kappa_\beta T_\beta + \vartheta\tau_\beta B_\beta\big) - \vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'T_\beta \\
&\quad -\big(\vartheta\kappa_\beta\big)^2 N_\beta + \big(\vartheta\tau_\beta\big)'B_\beta - \big(\vartheta\tau_\beta\big)^2 N_\beta\Big) \\
&= \Big(3\big(\vartheta\kappa_\beta\big)'\vartheta\kappa_\beta\Big)T_\beta + \Big(\big(\vartheta\kappa_\beta\big)^3 + \vartheta\kappa_\beta(\vartheta\tau_\beta)^2 - \big(\vartheta\kappa_\beta\big)''\Big)N_\beta \\
&\quad -\Big(2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)B_\beta
\end{aligned}$$

şeklinde bulunur. $\Delta H_\beta = 0$ ise

$$(\vartheta\kappa_\beta)'\vartheta\kappa_\beta = 0,\quad (\vartheta\kappa_\beta)^3 + \vartheta\kappa_\beta(\vartheta\tau_\beta)^2 - (\vartheta\kappa_\beta)'' = 0 \ \text{ ve } \ 2(\vartheta\kappa_\beta)'\vartheta\tau_\beta + \vartheta\kappa_\beta(\vartheta\tau_\beta)' = 0$$

birinci önerme sağlanır. $\Delta H_\beta = \lambda_0 H_\beta$ ise

$$(\vartheta\kappa_\beta)'\vartheta\kappa_\beta = 0,\quad (\vartheta\kappa_\beta)^3 + \vartheta\kappa_\beta(\vartheta\tau_\beta)^2 - (\vartheta\kappa_\beta)'' = \lambda_0\vartheta\kappa_\beta \quad \text{ve} \quad 2(\vartheta\kappa_\beta)'\vartheta\tau_\beta + \vartheta\kappa_\beta(\vartheta\tau_\beta)' = 0$$

ikinci önerme sağlanır ve bu da ispatı tamamlar. □

Sonuç 4.1.3 β eğrisi α nın bir involütü olsun. Eğer α bir dairesel helis ise, o zaman involütü β, konneksiyona göre 1. tipten harmoniktir.

İspat. α eğrisi bir dairesel helis olsun. Bu durumda involüt eğrisi β nın eğrilikleri $\kappa_\beta = \sqrt{2}/(c-s)$ ve $\tau_\beta = 0$ dır. Teorem 4.1.18 den
$(\vartheta\kappa_\beta)'\vartheta\kappa_\beta = 0, \quad (\vartheta\kappa_\beta)^3 + \vartheta\kappa_\beta(\vartheta\tau_\beta)^2 - (\vartheta\kappa_\beta)'' = \lambda_0\vartheta\kappa_\beta$ ve $-2(\vartheta\kappa_\beta)'\vartheta\tau_\beta - \vartheta\kappa_\beta(\vartheta\tau_\beta)' = 0$
eşitlikleri oluşur ve $\lambda_0 = 1$ elde edilir. Böylece ispat tamamlanır. $\square$

Teorem 4.1.19 β eğrisi α nın bir involütü olsun. β involüt eğrisinin harmoniklik koşulları, α eğrisinin eğrilikleri türünden aşağıdaki önermelerle verilir:

1. β involüt eğrisinin biharmonik olması için gerek ve yeter koşul

$$\frac{\kappa\left(\sqrt{\kappa^2+\tau^2}\right)''}{\sqrt{\kappa^2+\tau^2}} - \kappa\left(\kappa^2+\tau^2+\left(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\right)^2\right)$$

$$-2\tau\frac{\left(\kappa\tau'-\kappa'\tau\right)\sqrt{\kappa^2+\tau^2}'}{(\kappa^2+\tau^2)^{3/2}} - \tau\left(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\right)' = 0,$$

$$\tau\left(\kappa^2+\tau^2+\left(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\right)^2\right) - \frac{\tau\left(\sqrt{\kappa^2+\tau^2}\right)''}{\sqrt{\kappa^2+\tau^2}}$$

$$-2\kappa\frac{\left(\kappa\tau'-\kappa'\tau\right)\sqrt{\kappa^2+\tau^2}'}{(\kappa^2+\tau^2)^{3/2}} - \kappa\left(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\right)' = 0,$$

$$\kappa\kappa' + \tau\tau' = 0.$$

2. β involüt eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$\frac{\kappa\left(\sqrt{\kappa^2+\tau^2}\right)''}{\sqrt{\kappa^2+\tau^2}}-\kappa\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)$$

$$-2\tau\frac{\left(\kappa\tau'-\kappa'\tau\right)\sqrt{\kappa^2+\tau^2}'}{(\kappa^2+\tau^2)^{3/2}}-\tau\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)'=-\lambda\kappa,$$

$$\tau\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-\frac{\tau\left(\sqrt{\kappa^2+\tau^2}\right)''}{\sqrt{\kappa^2+\tau^2}}$$

$$-2\kappa\frac{\left(\kappa\tau'-\kappa'\tau\right)\sqrt{\kappa^2+\tau^2}'}{(\kappa^2+\tau^2)^{3/2}}-\kappa\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)'=\lambda\tau,$$

$$\kappa\kappa'+\tau\tau'=0,\quad \lambda_0\in\mathbb{R}.$$

İspat. H_β ortalama eğrilik vektörünün Laplace görüntüsü

$$\Delta H_\beta = 3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'T_\beta+\Big(\big(\vartheta\kappa_\beta\big)^3-\big(\vartheta\kappa_\beta\big)''+\vartheta^3\kappa_\beta\tau_\beta^2\Big)N_\beta$$

$$-\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'+\vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)B_\beta$$

dir. Bu ifade (3.1.7) ve (3.1.8) bağıntılarına göre α eğrisinin Frenet aparatları cinsinden yazılırsa $\Delta H_\beta=\Delta D_N N$ olur ve buradan

$$\begin{aligned}\Delta D_N N = {} & \Big(\frac{\kappa\left(\sqrt{\kappa^2+\tau^2}\right)''}{\sqrt{\kappa^2+\tau^2}}-\kappa\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-2\tau\frac{\left(\kappa\tau'-\kappa'\tau\right)\sqrt{\kappa^2+\tau^2}'}{(\kappa^2+\tau^2)^{3/2}}\\ & -\tau\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Big)T+\Big(\big(\sqrt{\kappa^2+\tau^2}\big)'\sqrt{\kappa^2+\tau^2}\Big)N+\Big(-\kappa\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\\ & +\tau\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-\frac{\tau\left(\sqrt{\kappa^2+\tau^2}\right)''}{\sqrt{\kappa^2+\tau^2}}-2\kappa\frac{\left(\kappa\tau'-\kappa'\tau\right)\sqrt{\kappa^2+\tau^2}'}{(\kappa^2+\tau^2)^{3/2}}\Big)B\end{aligned}$$

elde edilir. $\Delta D_N N = 0$ ise teoremin birinci önermesi, $\Delta D_N N = \lambda_0 D_N N$ ise teoremin ikinci önermesi sağlanır. $\square$

Sonuç 4.1.4 β eğrisi α nın bir involütü olsun. α eğrisi genel helis ise, o zaman involütü β eğrisi konneksiyona göre biharmoniktir.

İspat. α eğrisi genel helis olsun yani, $\tau/\kappa = sbt$. Bu eşitliğin türevi alınırsa

$$\kappa\tau' - \kappa'\tau = 0 \Longrightarrow \frac{\kappa}{\tau} = \frac{\kappa'}{\tau'} = sbt$$

dir. Diğer taraftan Teorem 4.1.19 da $\Delta D_N N = \lambda_0 D_N N$ koşuluna göre

$$\kappa\Big(\frac{\big(\sqrt{\kappa^2+\tau^2}\big)''}{\sqrt{\kappa^2+\tau^2}} - \big(\kappa^2+\tau^2\big)\Big) = 0,$$

$$-\tau\Big(\frac{\big(\sqrt{\kappa^2+\tau^2}\big)''}{\sqrt{\kappa^2+\tau^2}} - \big(\kappa^2+\tau^2\big)\Big) = 0,$$

$$\kappa\kappa' + \tau\tau' = 0$$

denklem sistemi meydana gelir. Buradan

$$\big(\kappa-\tau\big)\Big(\frac{\big(\sqrt{\kappa^2+\tau^2}\big)''}{\sqrt{\kappa^2+\tau^2}} - \big(\kappa^2+\tau^2\big)\Big) = 0$$

olup, $\kappa/\tau = sbt$ elde edilir ve böylece ispat tamamlanır. $\square$

Sonuç 4.1.5 β eğrisi α nın bir involütü olsun. α eğrisi dairesel helis ise, o zaman involütü β eğrisi, konneksiyona göre 1. tipten harmonik bir eğridir.

İspat. α eğrisi dairesel helis ise $\kappa = sbt$ ve $\tau = sbt$ olur. Teorem 4.1.19 dan $-\kappa\big(\kappa^2+\tau^2\big) = -\lambda_0\kappa$ ve $\tau\big(\kappa^2+\tau^2\big) = \lambda_0\tau$ eşitlikleri elde edilir.
$\Delta D_N N = \lambda_0 D_N N$ koşuluna göre β involüt eğrisinin $\lambda_0 = \kappa^2+\tau^2$ için 1.tipten harmonik olduğu açıktır. $\square$

Teorem 4.1.20 β eğrisi α nın bir involütü olsun. β involüt eğrisinin normal konneksiyona göre harmoniklik koşulları aşağıdaki önermelerle verilir:

1. β involüt eğrisinin zayıf biharmonik olması için gerek ve yeter koşul

$$\vartheta^3\kappa_\beta\tau_\beta^2 - \big(\vartheta\kappa_\beta\big)'' = 0 \quad \text{ve} \quad 2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)' = 0.$$

2. β involüt eğrisinin 1. tipten harmonik olması için gerek ve yeter koşul

$$\vartheta^3\kappa_\beta\tau_\beta^2 - \big(\vartheta\kappa_\beta\big)'' = \lambda_0\vartheta\kappa_\beta \quad \text{ve} \quad 2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)' = 0.$$

İspat. H_β ortalama eğrilik vektörünün normal Laplace görüntüsü

$$\Delta^\perp H_\beta^\perp = \Big(\vartheta^3\kappa_\beta\tau_\beta^2 - \big(\vartheta\kappa_\beta\big)''\Big)N_\beta - \Big(2\big(\vartheta\kappa_\beta\big)'\vartheta\tau_\beta + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big)B_\beta$$

şeklinde yazılır. $\Delta^\perp H_\beta^\perp = 0$ ise teoremin birinci önermesi, $\Delta^\perp H_\beta^\perp = \lambda_0 H_\beta^\perp$ ise teoremin ikinci önermesi sağlanır. □

Sonuç 4.1.6 β eğrisi α nın bir involütü olsun. α eğrisi dairesel helis ise, o zaman involütü β normal konneksiyona göre zayıf biharmoniktir.

İspat. α eğrisi dairesel helis olsun. İnvolüt eğrisi β nın eğrilik ve torsiyonu sırasıyla $\kappa_\beta = \sqrt{2}/(c-s)$ ve $\tau_\beta = 0$ dır. $\vartheta^3\kappa_\beta\tau_\beta^2 - (\vartheta\kappa_\beta)'' = 0$ ve $2(\vartheta\kappa_\beta)'\vartheta\tau_\beta + \vartheta\kappa_\beta(\vartheta\tau_\beta)' = 0$ eşitliklerinden teoremin birinci önermesi sağlanır. □

Örnek 4.1.3 $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ eğrisi verilsin. α eğrisinin Frenet vektörleri

$$T = \frac{1}{\sqrt{2}}(-sins, coss, 1), \quad N = (-coss, -sins, 0), \quad B = \frac{1}{\sqrt{2}}(sins, -coss, 1)$$

eğrilikleri de sırasıyla, $\kappa = \frac{1}{\sqrt{2}}$ ve $\tau = \frac{1}{\sqrt{2}}$ olur. α nın Frenet formülleri

$$D_TT = \frac{-1}{\sqrt{2}}(coss, sins, 0), \quad D_TN = (sins, -coss, 0), \quad D_TB = \frac{1}{\sqrt{2}}(coss, sins, 0)$$

ve Darboux vektörü $W = \frac{1}{\sqrt{2}}(T+B)$ olur. W ile B arasındaki açının türevi

$$\phi' = \frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2} = 0$$

bulunur. Buna göre β involüt eğrisi

$$\beta(s) = \frac{1}{\sqrt{2}}(coss - (c-s)sins,\ sins + (c-s)coss,\ c)\,, \quad c \in \mathbb{R} \text{ dir.}$$

β eğrisinin Frenet elemanları

$$T_\beta = (-coss, -sins, 0), \quad N_\beta = (sins, -coss, 0) \quad \text{ve} \quad B_\beta = (0,0,1)$$

şeklinde bulunur. İnvolüt eğrisi β nın eğrilikleri $\kappa_\beta = \frac{\sqrt{2}}{c-s}$, $\tau_\beta = 0$. W_β Darboux vektörü

$W_\beta = \frac{\sqrt{2}}{c-s} B_\beta = \frac{\sqrt{2}}{\lambda} W$ ve C_β birim Darboux vektörü $C_\beta = C$ olur.

Bu verilere göre β involüt eğrisini karakterize eden diferensiyel denklemleri aşağıdaki gibi vermek mümkündür:

1. (4.0.1) den W_β ye göre ve (4.1.20) den W ye göre,

2. (4.0.2) den C_β ye göre ve (4.1.22) den C ye göre,

3. (4.0.3) ten $W_\beta^\perp$ ye göre ve (4.1.24) ten $W^\perp$ ye göre,

4. (4.0.4) ten $C_\beta^\perp$ ye göre ve (4.1.26) dan $C^\perp$ ye göre,

5. (4.1.15) ten H_β ya göre ve (4.1.16) dan $D_N N$ ye göre,

6. (4.1.17) den $H_\beta^\perp$ ya göre ifade edilebilir.

1. durum: (4.0.1) den W_β ye göre:

$$\begin{aligned} W_\beta &= \tau_\beta T_\beta + \kappa_\beta B_\beta \\ &= \frac{\sqrt{2}}{\lambda} B_\beta\,, \\ D_{T_\beta} W_\beta &= \tau'_\beta T_\beta + \kappa'_\beta B_\beta \\ &= (\frac{\sqrt{2}}{\lambda})' B_\beta \end{aligned}$$

olduğundan diferensiyel denklem

$$D_{T_\beta} W_\beta + \frac{\lambda'}{\lambda} W_\beta = 0$$

olur. β eğrisinin denklemi (4.1.20) den W ye göre yazılırsa,

$$D_{T_\beta} W_\beta = D_N\big(\frac{\sqrt{2}}{\lambda} W\big)$$

eşitliğinden

$$D_N\big(\frac{1}{\lambda} W\big) - \frac{1}{\lambda} D_N W - \big(\frac{1}{\lambda}\big)' W = 0$$

elde edilir.

2. durum: (4.0.2) den C_β ye göre:

$$sin\phi_\beta = \frac{\tau_\beta}{\sqrt{\kappa_\beta^2 + \tau_\beta^2}} = 0 \quad \text{ve} \quad cos\phi_\beta = \frac{\kappa_\beta}{\sqrt{\kappa_\beta^2 + \tau_\beta^2}} = 1 \quad \text{olduğundan}$$

$$C_\beta = sin\phi_\beta T_\beta + cos\phi_\beta B_\beta \quad \Longrightarrow \quad C_\beta = B_\beta \ \text{olur.}$$

C_β nın T_β yönünde türevi alınırsa

$$\begin{aligned} D_{T_\beta} C_\beta &= \phi'\big(cos\phi_\beta T_\beta - sin\phi_\beta B_\beta\big) \\ D_{T_\beta} C_\beta &= 0 \end{aligned}$$

bulunur. β eğrisinin denklemi (4.1.22) den C ye göre:

$$\begin{aligned} C_\beta &= \frac{\phi'}{\sqrt{(\phi')^2 + \kappa^2 + \tau^2}} N + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{(\phi')^2 + \kappa^2 + \tau^2}} C \\ &= C \end{aligned}$$

olur. N yönünde türev alınırsa

$$\begin{aligned} D_N\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}N+\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}C\Big) &= 0 \\ D_N C &= 0 \end{aligned}$$

şeklinde elde edilir.

3. durum: (4.0.3) ten $W_\beta^\perp$ ye göre:

$W_\beta^\perp=\frac{\sqrt{2}}{\lambda}B_\beta \implies B_\beta=\frac{\lambda}{\sqrt{2}}W_\beta^\perp$ olur. $W_\beta^\perp$ in T_β yönünde normal konneksiyona göre

türevi alınırsa $D_{T_\beta}^\perp W_\beta^\perp=\big(\frac{\sqrt{2}}{\lambda}\big)'B_\beta$ olur ve diferensiyel denklem

$$D_{T_\beta}^\perp W_\beta^\perp+\big(\frac{\lambda'}{\lambda}\big)W_\beta^\perp=0$$

şeklinde elde edilir. β eğrisinin denklemi (4.1.24) ten $W^\perp$ ye göre:

$W_\beta^\perp=\frac{\sqrt{2}}{\lambda}W$ ifadesinin N yönünde normal konneksiyona göre türevi alınırsa

$$D_N^\perp\big(\frac{\sqrt{2}}{\lambda}W\big)=\big(\frac{\sqrt{2}}{\lambda}\big)'W+\frac{\sqrt{2}}{\lambda}D_N^\perp W$$

olur ve buradan

$$D_N^\perp\big(\frac{1}{\lambda}W\big)-\frac{1}{\lambda}D_N^\perp W-\big(\frac{1}{\lambda}\big)'W=0$$

elde edilir.

4. durum: (4.0.4) ten $C_\beta^\perp$ ye göre:

$C_\beta^\perp=B_\beta$ ifadesinin T_β yönünde normal konneksiyona göre türevi alınırsa

$$\begin{aligned} D_{T_\beta}^\perp C_\beta^\perp &= -\vartheta\tau_\beta cos\phi_\beta N_\beta-\phi_\beta' sin\phi_\beta B_\beta \\ D_{T_\beta}^\perp C_\beta^\perp &= 0 \end{aligned}$$

elde edilir. β eğrisinin denklemi (4.1.26) dan $C^{\perp}$ ye göre:

$$\begin{aligned} C_{\beta}^{\perp} &= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}C \\ &= C \end{aligned}$$

olur. N yönünde normal konneksiyona göre türev alınırsa

$$D_N^{\perp}C = 0$$

elde edilir.

5. durum: (4.1.15) ten H_β ya göre: $H_\beta = N_\beta$ dir. Bu ifadenin T_β yönünde konneksiyona göre birinci ve ikinci türevi alınırsa diferensiyel denklem

$$\begin{aligned} D_{T_\beta}H_\beta &= -T_\beta, \\ D_{T_\beta}^2 H_\beta &= -H_\beta, \\ &\Longrightarrow D_{T_\beta}^2 H_\beta + H_\beta = 0 \end{aligned}$$

şeklinde elde edilir. β eğrisinin denklemi (4.1.16) dan $D_N N$ ye göre:

$$H_\beta = D_N N$$

olur. Bu ifadenin N yönünde konneksiyona göre birinci ve ikinci türevi alınırsa

$$\begin{aligned} D_N^2 N &= D_N\big(D_N N\big) \\ &= -\big(\kappa^2+\tau^2\big)N, \\ D_N^3 N &= D_N\big(D_N^2 N\big) \\ &= -\big(\kappa^2+\tau^2\big)D_N N, \end{aligned}$$

ve buradan $\quad D_N^3 N + D_N N = 0 \quad$ elde edilir.

6. durum: (4.1.17) den $H_\beta^\perp$ ya göre:

$H_\beta^\perp = N_\beta$ dir. Bu ifadenin T_β yönünde normal konneksiyona göre türevi alınırsa

$$D_{T_\beta}^\perp H_\beta^\perp = 0$$

elde edilir.

4.2 Bertrand Partner Eğrisinin Karakterizasyonları

4.2.1 Bertrand Partner Eğrisinin Konneksiyona Göre Frenet Formülleri

Bu kısımda (α, γ) Bertrand eğri çifti olsun denildiğinde, Frenet elemanları $\{T,\ N,\ B,\ \kappa,\ \tau\}$ olan birim hızlı α Bertrand eğrisi ile Frenet elemanları $\{T_\gamma,\ N_\gamma,\ B_\gamma,\ \kappa_\gamma,\ \tau_\gamma\}$ olan γ partner eğrisi anlaşılacaktır.

Teorem 4.2.1 (α, γ) Bertrand eğri çifti olsun. Konneksiyona göre α eğrisinin Frenet vektörlerinin, B binormal vektörü yönünde türevleri

$$\begin{aligned}
D_B T &= \big(\frac{1-cos\theta}{sin\theta}\kappa\big)N, \\
D_B N &= \big(\frac{cos\theta-1}{sin\theta}\kappa\big)T + \big(\frac{1-cos\theta}{sin\theta}\tau\big)B, \\
D_B B &= \big(\frac{cos\theta-1}{sin\theta}\tau\big)N
\end{aligned} \tag{4.2.1}$$

bağıntısıyla verilir.

İspat. (3.1.4) bağıntısından $D_{T_\gamma}T_\gamma = \vartheta\kappa_\gamma N_\gamma$ dir. (3.1.11) ve (3.1.12) bağıntılarına göre bu eşitliğin sağ tarafı ve sol tarafı sırasıyla

$$\begin{aligned} D_{T_\gamma}T_\gamma &= \vartheta\kappa_\gamma N_\gamma \\ &= \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}N, \end{aligned} \tag{4.2.2}$$

$$\begin{aligned} D_{T_\gamma}T_\gamma &= D_{(cos\theta T + sin\theta B)}(cos\theta T + sin\theta B) \\ &= cos\theta D_T(cos\theta T + sin\theta B) + sin\theta D_B(cos\theta T + sin\theta B) \\ &= cos^2\theta\kappa N - cos\theta sin\theta\tau N + cos\theta sin\theta D_B T + sin^2\theta D_B B \end{aligned} \tag{4.2.3}$$

şeklinde bulunur. (4.2.2) ve (4.2.3) bağıntılarından

$$cos\theta sin\theta D_B T + sin^2\theta D_B B = \big(\vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} - \kappa cos^2\theta + \tau cos\theta sin\theta\big)N \tag{4.2.4}$$

yazılır. Benzer şekilde $D_{T_\gamma}N_\gamma$ vektörünün sağ tarafı ve sol tarafı sırasıyla

$$\begin{aligned} D_{T_\gamma}N_\gamma &= -\vartheta\kappa_\gamma T_\gamma + \vartheta\tau_\gamma B_\gamma \\ &= -\vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}\big(cos\theta T + sin\theta B\big) + \frac{\vartheta sin^2\theta}{\lambda^2\tau}\big(-sin\theta T + cos\theta B\big) \\ &= \big(\frac{-\vartheta sin^3\theta}{\lambda^2\tau} - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}cos\theta\big)T + \big(\frac{\vartheta sin^2\theta cos\theta}{\lambda^2\tau} - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}sin\theta\big)B, \end{aligned} \tag{4.2.5}$$

$$
\begin{aligned}
D_{T_\gamma}N_\gamma &= D_{(cos\theta T+sin\theta B)}N \\
&= cos\theta D_T N + sin\theta D_B N \\
&= -\kappa cos\theta T + \tau cos\theta B + sin\theta D_B N
\end{aligned} \tag{4.2.6}
$$

şeklinde bulunur. (4.2.5) ve (4.2.6) bağıntılarından

$$
\begin{aligned}
D_B N &= \Big(\kappa cot\theta - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}cot\theta - \frac{\vartheta sin^2\theta}{\lambda^2\tau}\Big)T \\
&\quad +\Big(\frac{\vartheta sin\theta cos\theta}{\lambda^2\tau} - \tau cot\theta - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}\Big)B
\end{aligned}
$$

olur. (3.1.13) bağıntısından $\lambda\kappa + \mu\tau = 1$ ve $\vartheta = \tau\sqrt{\lambda^2+\mu^2}$ ifadeleri burada yerine yazılırsa,

$$
D_B N = \big(\frac{cos\theta - 1}{sin\theta}\kappa\big)T + \big(\frac{1-cos\theta}{sin\theta}\tau\big)B
$$

bulunur. Son olarak benzer yöntem ve tekniklerle $D_{T_\gamma}B_\gamma$ vektörünün sağ tarafı ve sol tarafı sırasıyla

$$
\begin{aligned}
D_{T_\gamma}B_\gamma &= -\vartheta\tau_\gamma N_\gamma \\
&= -\frac{\vartheta sin^2\theta}{\lambda^2\tau}N,
\end{aligned} \tag{4.2.7}
$$

$$
\begin{aligned}
D_{T_\gamma}B_\gamma &= D_{(cos\theta T+sin\theta B)}(-sin\theta T+cos\theta B)\\
&= cos\theta D_T(-sin\theta T+cos\theta B)+sin\theta D_B(-sin\theta T+cos\theta B)\\
&= -cos\theta sin\theta D_T T+cos^2\theta D_T B-sin^2\theta D_B T+sin\theta cos\theta D_B B\\
&= (-\kappa cos\theta sin\theta-\tau cos^2\theta)N-sin^2\theta D_B T+sin\theta cos\theta D_B B \qquad (4.2.8)
\end{aligned}
$$

şeklinde olur. (4.2.7) ve (4.2.8) bağıntılarından

$$
-sin^2\theta D_B T+cos\theta sin\theta D_B B=(\kappa sin\theta cos\theta-\frac{\vartheta sin^2\theta}{\lambda^2\tau}+\tau cos^2\theta)N \qquad (4.2.9)
$$

bulunur. (4.2.4) ve (4.2.9) bağıntılarından $D_B T$ ve $D_B B$ hesaplanırsa,

$$
D_B T=\Big(\frac{\vartheta sin^2\theta}{\lambda^2\tau}+\vartheta\frac{\lambda\kappa-sin^2\theta}{\lambda(1-\lambda\kappa)}cot\theta-\kappa cot\theta\Big)N
$$

ve

$$
D_B B=\Big(\vartheta\frac{\lambda\kappa-sin^2\theta}{\lambda(1-\lambda\kappa)}-\frac{\vartheta sin\theta cos\theta}{\lambda^2\tau}+\tau cot\theta\Big)N
$$

olur. $\lambda\kappa+\mu\tau=1$ ve $\vartheta=\tau\sqrt{\lambda^2+\mu^2}$ ifadeleri yukarıda yerine yazılırsa,

$$
D_B T = \big(\frac{1-cos\theta}{sin\theta}\kappa\big)N
$$

ve

$$
D_B B = \big(\frac{cos\theta-1}{sin\theta}\tau\big)N
$$

elde edilir. □

Teorem 4.2.2 (α, γ) Bertrand eğri çifti olsun. Normal konneksiyona göre α eğrisinin Frenet vektörlerinin B binormal vektörü yönünde türevleri

$$D_B^{\perp}T = \big(\frac{1-cos\theta}{sin\theta}\kappa\big)N\,, \quad D_B^{\perp}N = \big(\frac{cos\theta-1}{sin\theta}\kappa\big)T \tag{4.2.10}$$

bağıntısıyla verilir.

İspat. (3.1.4) bağıntısından $D_{T_\gamma}N_\gamma = -\vartheta\kappa_\gamma T_\gamma + \vartheta\tau_\gamma B_\gamma$ dir. (3.1.11) ve (3.1.12) bağıntılarına göre bu ifadenin sağ tarafı

$$\begin{aligned} D_{T_\gamma}N_\gamma &= -\vartheta\kappa_\gamma T_\gamma + \vartheta\tau_\gamma B_\gamma \\ &= -\vartheta\kappa_\gamma(cos\theta T + sin\theta B) + \frac{\vartheta sin^2\theta}{\lambda^2\tau}(-sin\theta T + cos\theta B) \\ &= \big(\frac{-\vartheta sin^3\theta}{\lambda^2\tau} - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}cos\theta\big)T + \big(\frac{\vartheta sin^2\theta cos\theta}{\lambda^2\tau} - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}sin\theta\big)B \end{aligned} \tag{4.2.11}$$

ve sol tarafı

$$\begin{aligned} D_{T_\gamma}N_\gamma &= D_{(cos\theta T + sin\theta B)}N \\ &= cos\theta D_T N + sin\theta D_B N \\ &= -\kappa cos\theta T + \tau cos\theta B + sin\theta D_B N \end{aligned} \tag{4.2.12}$$

olur. (4.2.11) ve (4.2.12) bağıntılarından

$$\begin{aligned} D_B N &= \Big(\kappa cot\theta - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}cot\theta - \frac{\vartheta sin^2\theta}{\lambda^2\tau}\Big)T \\ &\quad + \Big(\frac{\vartheta sin\theta cos\theta}{\lambda^2\tau} - \tau cot\theta - \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}\Big)B \end{aligned} \tag{4.2.13}$$

bulunur. (3.2.4) bağıntısından $D_B^{\perp}N$ vektörü

$$D_B^{\perp}N \;=\; \big(\frac{cos\theta - 1}{sin\theta}\kappa\big)T$$

olur. Benzer yöntem ve tekniklerle $D_{T_\gamma}T_\gamma$ vektörünün sağ tarafı

$$D_{T_\gamma}T_\gamma \;=\; \vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}N \tag{4.2.14}$$

ve sol tarafı

$$D_{T_\gamma}T_\gamma \;=\; D_{(cos\theta T + sin\theta B)}(cos\theta T + sin\theta B)$$

$$\;=\; cos^2\theta\kappa N - cos\theta sin\theta\tau N + cos\theta sin\theta D_B T + sin^2\theta D_B B \tag{4.2.15}$$

şeklinde bulunur. (4.2.14) ve (4.2.15) bağıntılarından

$$cos\theta sin\theta D_B T + sin^2\theta D_B B \;=\; \big(\vartheta\frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} - \kappa cos^2\theta + \tau cos\theta sin\theta\big)N \tag{4.2.16}$$

yazılır. Benzer şekilde $D_{T_\gamma}B_\gamma$ vektörünün sağ tarafı

$$D_{T_\gamma}B_\gamma \;=\; -\frac{\vartheta sin^2\theta}{\lambda^2\tau}N \tag{4.2.17}$$

ve sol tarafı

$$D_{T_\gamma}B_\gamma \;=\; D_{(cos\theta T + sin\theta B)}(-sin\theta T + cos\theta B)$$

$$\;=\; (-\kappa cos\theta sin\theta - \tau cos^2\theta)N - sin^2\theta D_B T - sin\theta cos\theta D_B B \tag{4.2.18}$$

şeklinde yazılır. (4.2.17) ve (4.2.18) bağıntılarından

$$-sin^2\theta D_B T + cos\theta sin\theta D_B B = (\tau cos^2\theta - \frac{\vartheta sin^2\theta}{\lambda^2\tau} + \kappa sin\theta cos\theta)N \tag{4.2.19}$$

bulunur. (4.2.16) ve (4.2.19) bağıntılarından $D_B T$ hesaplanırsa

$$D_B T = \Big(-\kappa cot\theta + \frac{\vartheta sin^2\theta}{\lambda^2 \tau} + \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)} cot\theta \Big) N \tag{4.2.20}$$

olur ve (3.2.4) bağıntısından $D_B^{\perp} T$ vektörü

$$D_B^{\perp} T \quad = \quad \big(\frac{1-cos\theta}{sin\theta} \kappa \big) N$$

şeklinde elde edilir. $\square$

4.2.2 Bertrand Partner Eğrisinin Konneksiyona Göre Harmonikliği

Bu kısımda Bertrand eğri çiftlerinin konneksiyona göre biharmonik ve 1.tipten harmonik olma koşulları ile bu koşulların esas eğrinin Frenet aparatları türünden ifadeleri verildi.

Teorem 4.2.3 (α, γ) Bertrand eğri çifti ve γ partner eğrisinin ortalama eğrilik vektörü H_γ olsun. Buna göre aşağıdaki önermeler vardır:

1) γ partner eğrisinin biharmonik olması için gerek ve yeter koşul

$$3(\vartheta\kappa_\gamma)'\vartheta\kappa_\gamma = 0\,, \quad (\vartheta\kappa_\gamma)^3 + \vartheta\kappa_\gamma(\vartheta\tau_\gamma)^2 - (\vartheta\kappa_\gamma)'' = 0 \;\text{ ve }\; -2(\vartheta\kappa_\gamma)'\vartheta\tau_\gamma - \vartheta\kappa_\gamma(\vartheta\tau_\gamma)' = 0.$$

2) γ partner eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$3(\vartheta\kappa_\gamma)'\vartheta\kappa_\gamma = 0 \quad (\vartheta\kappa_\gamma)^3 + \vartheta\kappa_\gamma(\vartheta\tau_\gamma)^2 - (\vartheta\kappa_\gamma)'' = \lambda_0\vartheta\kappa_\gamma\,, \; -2(\vartheta\kappa_\gamma)'\vartheta\tau_\gamma - \vartheta\kappa_\gamma(\vartheta\tau_\gamma)' = 0.$$

İspat. γ partner eğrisinin ortalama eğrilik vektörü $H_\gamma = \vartheta\kappa_\gamma N_\gamma$ olsun. Bu vektörün Laplace görüntüsü

$$\begin{aligned}
\Delta H_\gamma &= -D^2_{T_\gamma}\big(\vartheta\kappa_\gamma N_\gamma\big) \\
&= -D_{T_\gamma}\big(D_{T_\gamma}(\vartheta\kappa_\gamma N_\gamma)\big) \\
&= -D_{T_\gamma}\Big(\big(\vartheta\kappa_\gamma\big)' N_\gamma + \big(\vartheta\kappa_\gamma\big) D_{T_\gamma} N_\gamma\Big) \\
&= -\big((\vartheta\kappa_\gamma)'' N_\gamma + 2(\vartheta\kappa_\gamma)' D_{T_\gamma} N_\gamma + \vartheta\kappa_\gamma D^2_{T_\gamma} N_\gamma\big) \\
&= -\Big(\big(\vartheta\kappa_\gamma\big)'' N_\gamma + 2\big(\vartheta\kappa_\gamma\big)'\big(-\vartheta\kappa_\gamma T_\gamma + \vartheta\tau_\gamma B_\gamma\big) + \vartheta\kappa_\gamma\big(-\vartheta\kappa_\gamma\big)' T_\gamma \\
&\quad -\big(\vartheta\kappa_\gamma\big)^2 N_\gamma + \big(\vartheta\tau_\gamma\big)' B_\gamma - \big(\vartheta\tau_\gamma\big)^2 N_\gamma\Big) \\
&= 3\big(\vartheta\kappa_\gamma\big)' \vartheta\kappa_\gamma T_\gamma + \Big(\big(\vartheta\kappa_\gamma\big)^3 + \vartheta\kappa_\gamma\big(\vartheta\tau_\gamma\big)^2 - \big(\vartheta\kappa_\gamma\big)''\Big) N_\gamma \\
&\quad -\Big(2\big(\vartheta\kappa_\gamma\big)'\vartheta\tau_\gamma + \vartheta\kappa_\gamma\big(\vartheta\tau_\gamma\big)'\Big) B_\gamma.
\end{aligned}$$

$\Delta H_\gamma = 0$ ise

$$(\vartheta\kappa_\gamma)'\vartheta\kappa_\gamma = 0, \quad (\vartheta\kappa_\gamma)^3 + \vartheta\kappa_\gamma(\vartheta\tau_\gamma)^2 - (\vartheta\kappa_\gamma)'' = 0 \ \text{ ve } \ 2(\vartheta\kappa_\gamma)'\vartheta\tau_\gamma + \vartheta\kappa_\gamma(\vartheta\tau_\gamma)' = 0$$

birinci önerme sağlanır.

$\Delta H_\gamma = \lambda_0 H_\gamma$ ise

$$(\vartheta\kappa_\gamma)'\vartheta\kappa_\gamma = 0, \quad (\vartheta\kappa_\gamma)^3 + \vartheta\kappa_\gamma(\vartheta\tau_\gamma)^2 - (\vartheta\kappa_\gamma)'' = \lambda_0\vartheta\kappa_\gamma \ \text{ ve } \ 2(\vartheta\kappa_\gamma)'\vartheta\tau_\gamma + \vartheta\kappa_\gamma(\vartheta\tau_\gamma)' = 0$$

ikinci önerme sağlanır. Bu da ispatı tamamlar. □

Teorem 4.2.4 (α, γ) Bertrand eğri çifti olsun. Konneksiyona göre γ partner eğrisinin harmoniklik koşulları, α eğrisinin Frenet aparatları türünden aşağıdaki bağıntı ile verilir:
1) γ partner eğrisinin biharmonik olması için gerek ve yeter koşul

$$(3\lambda\kappa - sin^2\theta)\kappa' = 0,$$

$$(\lambda\kappa - sin^2\theta)(1 - cos\theta)^2(\kappa^2 + \tau^2) - \lambda\kappa'' sin^2\theta = 0,$$

$$sin^2\theta\tau' - \lambda\kappa\tau' - 2\lambda\kappa'\tau = 0.$$

2) γ partner eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul,

$$(3\lambda\kappa - sin^2\theta)\kappa' = 0,$$

$$\frac{(\lambda\kappa - sin^2\theta)(1 - cos\theta)^2(\kappa^2 + \tau^2) - \lambda\kappa'' sin^2\theta}{sin^2\theta} = \lambda_0\,(\lambda\kappa - sin^2\theta),$$

$$sin^2\theta\tau' - \lambda\kappa\tau' - 2\lambda\kappa'\tau = 0.$$

İspat. (3.1.11) ve (3.1.12) bağıntılarınan H_γ ortalama eğrilik vektörü α eğrisinin Frenet aparatları türünden yazılırsa

$$H_\gamma = \vartheta \frac{\lambda\kappa - sin^2\theta}{\lambda(1 - \lambda\kappa)} N$$

olur. Bu eşitlikteki N vektörünün katsayısı b_1 ile gösterilirse, $H_\gamma = b_1 N$ yazılır. Diğer taraftan Teorem 4.2.1 deki Frenet formülünün vektörleri

$$\frac{1 - cos\theta}{sin\theta}\kappa N = \alpha_1 N\,, \qquad \frac{cos\theta - 1}{sin\theta}\tau N = \alpha_2 N$$

şeklinde ifade edililirse (3.1.1) ve (4.2.1) bağıntılarına göre H_γ nın Laplace görüntüsü,

$$
\begin{aligned}
\Delta H_\gamma &= -D_B^2(H_\gamma) \\
&= -D_B D_B(b_1 N) \\
&= -D_B(b_1' N + b_1 D_B N) \\
&= D_B(b_1\alpha_1 T - b_1' N + b_1\alpha_2 B) \\
&= (b_1\alpha_1)' T + b_1\alpha_1 D_B T - b_1'' N - b_1' D_B N + (b_1\alpha_2)' B + b_1\alpha_2 D_B B \\
&= \big(b_1'\alpha_1 + (b_1\alpha_1)'\big) T + \big(b_1(\alpha_1^2 + \alpha_2^2) - b_1''\big) N + \big(b_1'\alpha_2 + (b_1\alpha_2)'\big) B \\
&= \big(\frac{(3\lambda\kappa - sin^2\theta)\kappa'}{\mu(1+cos\theta)}\big) T + \big(\frac{(\lambda\kappa - sin^2\theta)(1-cos\theta)^2(\kappa^2+\tau^2) - \lambda\kappa'' sin^2\theta}{\mu sin^3\theta}\big) N \\
&\quad + \big(\frac{sin^2\theta\tau' - \lambda\kappa\tau' - 2\lambda\kappa'\tau}{\mu(1+cos\theta)}\big) B
\end{aligned}
$$

şeklinde bulunur. $\Delta H_\gamma = 0$ ise birinci önerme, $\Delta H_\gamma = \lambda_0 H_\gamma$ ise ikinci önerme sağlanır ve bu da ispatı tamamlar. $\square$

Teorem 4.2.5 (α, γ) Bertrand eğri çifti olsun. Normal konneksiyona göre γ partner eğrisinin harmoniklik koşulları, aşağıdaki bağıntı ile verilir:

1) γ eğrisinin zayıf biharmonik olması için gerek ve yeter koşul

$$\vartheta^3\tau_\gamma^2\kappa_\gamma - (\vartheta\kappa_\gamma)'' = 0 \quad \text{ve} \quad (\vartheta\kappa_\gamma)'\vartheta\tau_\gamma + (\vartheta^2\kappa_\gamma\tau_\gamma)' = 0\,,$$

2) γ eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$\vartheta^3\tau_\gamma^2\kappa_\gamma - (\vartheta\kappa_\gamma)'' = \lambda_0\kappa_\gamma \quad \text{ve} \quad (\vartheta\kappa_\gamma)'\vartheta\tau_\gamma + (\vartheta^2\kappa_\gamma\tau_\gamma)' = 0\,, \;\; \lambda_0 \in \mathbb{R} \text{ dir.}$$

İspat. (3.2.4) bağıntısından $\Delta^{\perp} H_\gamma = -D^{\perp}_{T_\gamma} D^{\perp}_{T_\gamma} H_\gamma$ yazılır. Buna göre $\Delta^{\perp} H_\gamma$ hesaplanırsa,

$$\begin{aligned}
D_{T_\gamma} H_\gamma &= D_{T_\gamma}\big(\vartheta\kappa_\gamma N_\gamma\big) \\
&= \big(\vartheta\kappa_\gamma\big)' N_\gamma + \vartheta\kappa_\gamma D_{T_\gamma} N_\gamma \\
&= \big(\vartheta\kappa_\gamma\big)' N_\gamma + \vartheta\kappa_\gamma\big(-\vartheta\kappa_\gamma T_\gamma + \vartheta\tau_\gamma B_\gamma\big) \\
&= \big(\vartheta\kappa_\gamma\big)' N_\gamma - \big(\vartheta\kappa_\gamma\big)^2 T_\gamma + \big(\vartheta^2\kappa_\gamma\tau_\gamma\big) B_\gamma
\end{aligned}$$

olur. Bu ifadenin normal tümleyeni

$$D^{\perp}_{T_\gamma} H_\gamma = \big(\vartheta\kappa_\gamma\big)' N_\gamma + \big(\vartheta^2\kappa_\gamma\tau_\gamma\big) B_\gamma$$

dir. Tekrar türev alınırsa,

$$\begin{aligned}
D_{T_\gamma}\big(D^{\perp}_{T_\gamma} H_\gamma\big) &= D_{T_\gamma}\big((\vartheta\kappa_\gamma)' N_\gamma + \vartheta^2\kappa_\gamma\tau_\gamma B_\gamma\big) \\
&= \big(\vartheta\kappa_\gamma\big)'' N_\gamma + \big(\vartheta\kappa_\gamma\big)' D_{T_\gamma} N_\gamma + \big(\vartheta^2\kappa_\gamma\tau_\gamma\big)' B_\gamma + \vartheta^2\kappa_\gamma\tau_\gamma D_{T_\gamma} B_\gamma \\
&= \big(\vartheta\kappa_\gamma\big)'' N_\gamma + \big(\vartheta\kappa_\gamma\big)'\big(-\vartheta\kappa_\gamma T_\gamma + \vartheta\tau_\gamma B_\gamma\big) + \big(\vartheta^2\kappa_\gamma\tau_\gamma\big)' B_\gamma \\
&\quad + \vartheta^2\kappa_\gamma\tau_\gamma\big(-\vartheta\tau_\gamma N_\gamma\big)
\end{aligned}$$

bulunur. Bu ifadenin normal tümleyeni

$$D^{\perp}_{T_\gamma} D^{\perp}_{T_\gamma} H_\gamma = \Big(\big(\vartheta\kappa_\gamma\big)'' - \vartheta^3\tau_\gamma^2\kappa_\gamma\Big) N_\gamma + \Big(\big(\vartheta\kappa_\gamma\big)'\vartheta\tau_\gamma + \big(\vartheta^2\kappa_\gamma\tau_\gamma\big)'\Big) B_\gamma$$

olur ve buradan

$$\Delta^{\perp} H_\gamma = \Big(\vartheta^3\tau_\gamma^2\kappa_\gamma - \big(\vartheta\kappa_\gamma\big)''\Big) N_\gamma - \Big(\big(\vartheta\kappa_\gamma\big)'\vartheta\tau_\gamma + \big(\vartheta^2\kappa_\gamma\tau_\gamma\big)'\Big) B_\gamma$$

elde edilir. $\Delta^{\perp}H_{\gamma}=0$ ise

$$\vartheta^3\tau_\gamma^2\kappa_\gamma-(\vartheta\kappa_\gamma)''=0 \quad \text{ve} \quad (\vartheta\kappa_\gamma)'\vartheta\tau_\gamma+(\vartheta^2\kappa_\gamma\tau_\gamma)'=0$$

birinci önerme, $\Delta^{\perp}H_{\gamma}=\lambda_0 H_{\gamma}$ ise

$$\vartheta^3\tau_\gamma^2\kappa_\gamma-(\vartheta\kappa_\gamma)''=\lambda_0\kappa_\gamma \quad \text{ve} \quad (\vartheta\kappa_\gamma)'\vartheta\tau_\gamma+(\vartheta^2\kappa_\gamma\tau_\gamma)'=0$$

ikinci önerme sağlanır. □

Teorem 4.2.6 (α,γ) Bertrand eğri çifti olsun. Normal konneksiyona göre γ partner eğrisinin harmoniklik koşulları, α eğrisinin Frenet aparatları türünden aşağıdaki bağıntı ile verilir:

1) γ eğrisinin zayıf biharmonik olması için gerek ve yeter koşul

$$3\lambda\kappa\kappa'-sin^2\theta\,\kappa'=0,$$

$$\frac{(\lambda\kappa-sin^2\theta)(1-cos\theta)^2\kappa^2-\lambda\kappa''sin^2\theta}{\mu sin^3\theta}=0.$$

2) γ eğrisinin 1.tipten harmonik olması için gerek ve yeter koşul

$$3\lambda\kappa\kappa'-sin^2\theta\,\kappa'=0,$$

$$\frac{(\lambda\kappa-sin^2\theta)(1-cos\theta)^2\kappa^2-\lambda\kappa''sin^2\theta}{sin^2\theta}=\lambda_0(\lambda\kappa-sin^2\theta).$$

İspat. (3.2.3) ve (4.2.10) bağıntısından normal Laplace dönüşümü hesaplanırsa,

$$\begin{aligned}\Delta^{\perp}H_{\gamma} &= -D_B^{\perp}D_B^{\perp}\big(b_1N\big)\\ &= -D_B^{\perp}\big(b_1'N+b_1D_B^{\perp}N\big)\end{aligned}$$

$$= D_B^{\perp}\big(b_1\alpha_1 T - b_1' N\big)$$

$$= \Big(b_1'\alpha_1 + \big(b_1\alpha_1\big)'\Big)T + \Big(b_1\alpha_1^2 - b_1''\Big)N$$

$$= \Big(\frac{3\lambda\kappa\kappa' - sin^2\theta\,\kappa'}{\mu(1+cos\theta)}\Big)T + \Big(\frac{(\lambda\kappa - sin^2\theta)(1-cos\theta)^2\kappa^2 - \lambda\kappa'' sin^2\theta}{\mu sin^3\theta}\Big)N$$

olur. $\Delta^{\perp}H_\gamma = 0$ ise birinci önerme, $\Delta^{\perp}H_\gamma = \lambda_0 H_\gamma$ ise ikinci önerme sağlanır. $\square$

Sonuç 4.2.1 Dairesel helis eğrisinin Bertrand partneri, konneksiyona göre 1.tipten harmonik bir eğridir.

İspat. $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ verilsin. α eğrisinin Frenet vektörleri

$$T = \frac{1}{\sqrt{2}}(-sins, coss, 1), \quad N = (-coss, -sins, 0), \quad B = \frac{1}{\sqrt{2}}(sins, -coss, 1)$$

ve eğrilikleri de sırasıyla, $\kappa = \frac{1}{\sqrt{2}}$, $\tau = \frac{1}{\sqrt{2}}$ dir. α eğrisinin Frenet formülleri

$$D_T T = \frac{-1}{\sqrt{2}}(coss, sins, 0), \qquad D_T N = (sins, -coss, 0), \qquad D_T B = \frac{1}{\sqrt{2}}(coss, sins, 0)$$

ve α nın Bertrand partner eğrisi ise,

$$\gamma(s) = \frac{1}{\sqrt{2}}(coss, sins, s) + \lambda(-coss, -sins, 0) \text{ dir.}$$

(4.2.1) bağıntısından

$$H_\gamma = \frac{\lambda\kappa - sin^2\theta}{\mu sin\theta}N$$

ifadesinin Laplace görüntüsü

$$\Delta H_\gamma = \frac{\lambda\kappa - sin^2\theta}{\mu\, sin\theta}\,\Big(\frac{cos\theta - 1}{sin\theta}\Big)^2 N$$

şeklinde bulunur. Böylece Teorem 4.2.4 ten γ partner eğrisi konneksiyona göre

$$\lambda_0 = \Big(\frac{cos\theta - 1}{sin\theta}\Big)^2$$

için 1.tipten harmoniktir. $\square$

Sonuç 4.2.2 Dairesel helis eğrisinin Bertrand partneri, normal konneksiyona göre 1.tipten harmonik bir eğridir.

İspat. $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ verilsin. α eğrisinin Frenet elemanları Sonuç 4.2.1 de hesaplandı. (4.2.10) bağıntısından ortalama eğrilik vektörü

$$H_\gamma = \frac{\lambda\kappa - sin^2\theta}{\mu sin\theta} N$$

nin normal Laplace görüntüsü

$$\Delta^\perp H_\gamma = \frac{\lambda\kappa - sin^2\theta}{\mu\, sin\theta} \Big(\frac{cos\theta - 1}{\sqrt{2}\, sin\theta}\Big)^2 N$$

dir. Böylece Teorem 4.2.6 dan γ partner eğrisi normal konneksiyona göre

$$\lambda_0 = \frac{1}{2}\Big(\frac{cos\theta - 1}{sin\theta}\Big)^2$$

için 1.tipten harmoniktir. □

4.2.3 Bertrand Partner Eğrisinin Konneksiyona Göre Denklemleri

Teorem 4.2.7 (α, γ) Bertrand eğri çifti olsun. γ partner eğrisinin T_γ teğet vektörüne göre diferensiyel denklemi

$$D^3_{T_\gamma} T_\gamma - \Big(3\frac{\vartheta'}{\vartheta} + 2\frac{\kappa'_\gamma}{\kappa_\gamma} + \frac{\tau'_\gamma}{\tau_\gamma}\Big) D^2_{T_\gamma} T_\gamma$$

$$+\Big(\vartheta^2\big(\kappa_\gamma^2 + \tau_\gamma^2\big) - \frac{\vartheta''}{\vartheta} - \frac{\kappa''_\gamma}{\kappa_\gamma} + \big(\frac{\vartheta'}{\vartheta} + \frac{\kappa'_\gamma}{\kappa_\gamma}\big)\big(3\frac{\vartheta'}{\vartheta} + \frac{\tau'_\gamma}{\tau_\gamma}\big) + 2\big(\frac{\kappa'_\gamma}{\kappa_\gamma}\big)^2\Big) D_{T_\gamma} T_\gamma$$

$$+\vartheta^2 \kappa_\gamma \tau_\gamma \big(\frac{\kappa_\gamma}{\tau_\gamma}\big)' T_\gamma = 0$$

şeklinde verilir.

İspat. (3.1.4) bağıntısından N_γ asli normal ve B_γ binormal vektörleri çekilirse

$$N_\gamma = \frac{1}{\vartheta\kappa_\gamma} D_{T_\gamma} T_\gamma \quad \text{ve} \quad B_\gamma = \frac{1}{\vartheta\tau_\gamma} D_{T_\gamma} N_\gamma + \frac{\kappa_\gamma}{\tau_\gamma} T_\gamma$$

olur. B_γ vektöründe N_γ nin eşiti yazılır ve ardından türev alınırsa

$$\begin{aligned}
B_\gamma &= \frac{1}{\vartheta\tau_\gamma} D_{T_\gamma}\big(\frac{1}{\vartheta\kappa_\gamma} D_{T_\gamma} T_\gamma\big) + \frac{\kappa_\gamma}{\tau_\gamma} T_\gamma \\
&= \frac{1}{\vartheta\tau_\gamma}\Big((\frac{1}{\vartheta\kappa_\gamma})' D_{T_\gamma} T_\gamma + \frac{1}{\vartheta\tau_\gamma}\frac{1}{\vartheta\kappa_\gamma} D^2_{T_\gamma} T_\gamma\Big) + \frac{\kappa_\gamma}{\tau_\gamma} T_\gamma \\
&= \frac{1}{\vartheta\tau_\gamma}(\frac{1}{\vartheta\kappa_\gamma})' D_{T_\gamma} T_\gamma + \frac{1}{\vartheta^2\tau_\gamma\kappa_\gamma} D^2_{T_\gamma} T_\gamma + \frac{\kappa_\gamma}{\tau_\gamma} T_\gamma\,,
\end{aligned}$$

$$D_{T_\gamma} B_\gamma = D_{T_\gamma}\Big(\frac{1}{\vartheta\tau_\gamma}(\frac{1}{\vartheta\kappa_\gamma})' D_{T_\gamma} T_\gamma\Big) + D_{T_\gamma}\Big(\frac{1}{\vartheta^2\kappa_\gamma\tau_\gamma} D^2_{T_\gamma} T_\gamma\Big) + D_{T_\gamma}\Big(\frac{\kappa_\gamma}{\tau_\gamma} T_\gamma\Big)$$

bulunur. Diğer taraftan

$$\begin{aligned}
D_{T_\gamma} B_\gamma &= -\vartheta\tau_\gamma \frac{1}{\vartheta\kappa_\gamma} D_{T_\gamma} T_\gamma \\
&= -\frac{\tau_\gamma}{\kappa_\gamma} D_{T_\gamma} T_\gamma
\end{aligned}$$

dir. Bu ifade üstteki eşitliğin sol tarafında yerine yazılırsa,

$$\begin{aligned}
-\frac{\tau_\gamma}{\kappa_\gamma} D_{T_\gamma} T_\gamma &= \Big(\frac{1}{\vartheta\tau_\gamma}(\frac{1}{\vartheta\kappa_\gamma})'\Big)' D_{T_\gamma} T_\gamma + \frac{1}{\vartheta\tau_\gamma}(\frac{1}{\vartheta\kappa_\gamma})' D^2_{T_\gamma} T_\gamma \\
&\quad + (\frac{1}{\vartheta^2\kappa_\gamma\tau_\gamma})' D^2_{T_\gamma} T_\gamma + \frac{1}{\vartheta^2\kappa_\gamma\tau_\gamma} D^3_{T_\gamma} T_\gamma + (\frac{\kappa_\gamma}{\tau_\gamma})' T_\gamma + \frac{\kappa_\gamma}{\tau_\gamma} D_{T_\gamma} T_\gamma
\end{aligned}$$

veya

$$\begin{aligned}
\frac{1}{\vartheta^2\kappa_\gamma\tau_\gamma} D^3_{T_\gamma} T_\gamma &+ \Big(\frac{1}{\vartheta\tau_\gamma}(\frac{1}{\vartheta\kappa_\gamma})' + (\frac{1}{\vartheta^2\kappa_\gamma\tau_\gamma})'\Big) D^2_{T_\gamma} T_\gamma \\
&+ \Big(\frac{\kappa_\gamma^2 + \tau_\gamma^2}{\kappa_\gamma\tau_\gamma} + (\frac{1}{\vartheta\tau_\gamma}(\frac{1}{\vartheta\kappa_\gamma})')'\Big) D_{T_\gamma} T_\gamma + (\frac{\kappa_\gamma}{\tau_\gamma})' T_\gamma = 0
\end{aligned}$$

elde edilir. Burada $D^2_{T_\gamma}T_\gamma$ ve $D_{T_\gamma}T_\gamma$ nin katsayıları sırasıyla,

$$\frac{1}{\vartheta\tau_\gamma}\big(\frac{1}{\vartheta\kappa_\gamma}\big)' + \big(\frac{1}{\vartheta^2\kappa_\gamma\tau_\gamma}\big)' = -\frac{\vartheta'\kappa_\gamma + \vartheta\kappa'_\gamma}{\vartheta^3\tau_\gamma\kappa^2_\gamma} - \frac{2\vartheta\vartheta'\kappa_\gamma\tau_\gamma + \vartheta^2\kappa'_\gamma\tau_\gamma + \vartheta^2\kappa_\gamma\tau'_\gamma}{\vartheta^4\kappa^2_\gamma\tau^2_\gamma},$$

$$\begin{aligned}\frac{\kappa_\gamma}{\tau_\gamma} + \frac{\tau_\gamma}{\kappa_\gamma} + \Big(\frac{1}{\vartheta\tau_\gamma}\big(\frac{1}{\vartheta\kappa_\gamma}\big)'\Big)' &= \frac{\kappa^2_\gamma + \tau^2_\gamma}{\kappa_\gamma\tau_\gamma} - \frac{\vartheta''\kappa_\gamma + 2\vartheta'\kappa'_\gamma + \vartheta\kappa''_\gamma}{\vartheta^3\tau_\gamma\kappa^2_\gamma} + \\ &\quad \frac{\big(3\vartheta^2\vartheta'\tau_\gamma\kappa^2_\gamma + \vartheta^3\tau'_\gamma\kappa^2_\gamma + 2\vartheta^3\tau_\gamma\kappa_\gamma\kappa'_\gamma\big)\big(\vartheta'\kappa_\gamma + \vartheta\kappa'_\gamma\big)}{\vartheta^6\tau^2_\gamma\kappa^4_\gamma}\end{aligned}$$

dır. $\square$

Teorem 4.2.8 (α,γ) Bertrand eğri çifti olsun. γ partner eğrisinin konneksiyona göre diferensiyel denklemi α eğrisinin B binormal vektörü yönünde türevler cinsinden

$$\begin{aligned}D^3_B B - \Big(\frac{\kappa'}{\kappa} + 2\frac{\tau'}{\tau}\Big)D^2_B B + \Big(\big(\frac{1-cos\theta}{sin\theta}\big)^2\big(\kappa^2+\tau^2\big) &+ 2\big(\frac{\tau'}{\tau}\big)^2 + \frac{\kappa'\tau'}{\kappa\tau} - \frac{\tau''}{\tau}\Big)D_B B \\ &+ \Big(\kappa\tau\big(\frac{1-cos\theta}{sin\theta}\big)^2\big(\frac{\tau}{\kappa}\big)'\Big)B = 0 \qquad (4.2.21)\end{aligned}$$

bağıntısıyla verilir.

İspat. (4.2.1) bağıntısında Frenet vektörlerinin katsayıları

$$\frac{1-cos\theta}{sin\theta}\kappa = \alpha_1 \qquad \text{ve} \qquad \frac{cos\theta - 1}{sin\theta}\tau = \alpha_2$$

şeklinde gösterilirse,

$$D_B T = \alpha_1 N\,, \quad D_B N = -\alpha_1 T - \alpha_2 B\,, \quad D_B B = \alpha_2 N$$

olur. N vektörünün eşiti

$$D_B B = \alpha_2 N \implies N = \frac{1}{\alpha_2} D_B B$$

dir. Diğer taraftan

$$D_B T = \alpha_1 N \quad \text{ve} \quad D_B B = \alpha_2 N \implies D_B T = \frac{\alpha_1}{\alpha_2} D_B B$$

yazılır. $D_B B = \alpha_2 N$ ifadesinin B yönünde birinci, ikinci ve üçüncü türevi alınırsa

$$\begin{aligned}
D_B B &= \alpha_2 N, \\
D_B\big(D_B B\big) &= D_B\big(\alpha_2 N\big) \\
D_B^2 B &= -\alpha_1\alpha_2 T + \alpha_2' N - \alpha_2^2 B, \\
D_B\big(D_B^2 B\big) &= D_B\big(-\alpha_1\alpha_2 T + \alpha_2' N - \alpha_2^2 B\big) \\
D_B^3 B &= \big(-\alpha_1\alpha_2\big)' T - \alpha_1\alpha_2 D_B T + \alpha_2'' N + \alpha_2' D_B N - \big(\alpha_2^2\big)' B - \alpha_2^2 D_B B
\end{aligned}$$

bulunur. Burada $D_B N$, $D_B T$ ve N vektörünün yerine eşitleri yazılırsa

$$D_B^3 B = \Big(-\big(\alpha_1\alpha_2\big)' - \alpha_1\alpha_2'\Big) T + \Big(\frac{\alpha_2''}{\alpha_2} - \alpha_1^2 - \alpha_2^2\Big) D_B B - \Big(\alpha_2\alpha_2' + \big(\alpha_2^2\big)'\Big) B$$

olur. Son olarak T vektörünün eşiti hesaplanırsa

$$\begin{aligned}
D_B N = -\alpha_1 T - \alpha_2 B \quad \implies \quad T &= -\frac{1}{\alpha_1} D_B N - \frac{\alpha_2}{\alpha_1} B \\
&= -\frac{1}{\alpha_1} D_B\big(\frac{1}{\alpha_2} D_B B\big) - \frac{\alpha_2}{\alpha_1} B \\
&= \frac{-1}{\alpha_1\alpha_2} D_B^2 B + \frac{\alpha_2'}{\alpha_1\alpha_2^2} D_B B - \frac{\alpha_2}{\alpha_1} B
\end{aligned}$$

bulunur. T nin eşiti yukarıda yerine yazılır ve düzenleme yapılırsa

$$D_B^3 B + \Big((\alpha_1\alpha_2)' + \alpha_1\alpha_2' \Big)\Big(\frac{-1}{\alpha_1\alpha_2} D_B^2 B + \frac{\alpha_2'}{\alpha_1\alpha_2^2} D_B B - \frac{\alpha_2}{\alpha_1} B \Big) \quad + \quad \Big(\frac{-\alpha_2''}{\alpha_2} + \alpha_1^2 + \alpha_2^2 \Big) D_B B$$

$$+ \Big(\alpha_2\alpha_2' + (\alpha_2^2)' \Big) B = 0$$

elde edilir. □

Teorem 4.2.9 (α, γ) Bertrand eğri çifti olsun. γ partner eğrisinin normal konneksiyona göre diferensiyel denklemi N_γ asli normali veya B_γ binormali türünden

$$D_{T_\gamma}^{\perp} D_{T_\gamma}^{\perp} N_\gamma - \frac{(\vartheta\tau_\gamma)'}{\vartheta\tau_\gamma} D_{T_\gamma}^{\perp} N_\gamma + (\vartheta\tau_\gamma)^2 N_\gamma = 0$$

veya

$$D_{T_\gamma}^{\perp} D_{T_\gamma}^{\perp} B_\gamma - \frac{(\vartheta\tau_\gamma)'}{\vartheta\tau_\gamma} D_{T_\gamma}^{\perp} B_\gamma + (\vartheta\tau_\gamma)^2 B_\gamma = 0$$

bağıntısıyla verilir.

İspat. (3.2.3) bağıntısından N_γ asli normal ve B_γ binormalin eşitleri çekilirse

$$N_\gamma = \frac{-1}{\vartheta\tau_\gamma} D_{T_\gamma}^{\perp} B_\gamma \quad \text{ve} \quad B_\gamma = \frac{1}{\vartheta\tau_\gamma} D_{T_\gamma}^{\perp} N_\gamma$$

olur. $D_{T_\gamma}^{\perp} N_\gamma = \vartheta\tau_\gamma B_\gamma$ ifadesinin normal konneksiyona göre türevi alınırsa

$$\begin{aligned}
D_{T_\gamma}^{\perp}\big(D_{T_\gamma}^{\perp} N_\gamma\big) &= D_{T_\gamma}^{\perp}\big(\vartheta\tau_\gamma B_\gamma\big) \\
&= \big(\vartheta\tau_\gamma\big)' B_\gamma + \vartheta\tau_\gamma D_{T_\gamma}^{\perp} B_\gamma \\
&= \big(\vartheta\tau_\gamma\big)' \Big(\frac{1}{\vartheta\tau_\gamma} D_{T_\gamma}^{\perp} N_\gamma \Big) + \vartheta\tau_\gamma \big(-\vartheta\tau_\gamma N_\gamma \big)
\end{aligned}$$

olur ve buradan

$$D_{T_\gamma}^{\perp} D_{T_\gamma}^{\perp} N_\gamma - \frac{(\vartheta\tau_\gamma)'}{\vartheta\tau_\gamma} D_{T_\gamma}^{\perp} N_\gamma + \big(\vartheta\tau_\gamma\big)^2 N_\gamma = 0$$

elde edilir. Benzer şekilde $D^{\perp}_{T_\gamma}B_\gamma = -\vartheta\tau_\gamma N_\gamma$ ifadesinin normal konneksiyona göre türevi alınırsa

$$\begin{aligned} D^{\perp}_{T_\gamma}\big(D^{\perp}_{T_\gamma}B_\gamma\big) &= D^{\perp}_{T_\gamma}\big(-\vartheta\tau_\gamma N_\gamma\big) \\ &= \big(\vartheta\tau_\gamma\big)' N_\gamma - \vartheta\tau_\gamma D^{\perp}_{T_\gamma}N_\gamma \\ &= \big(-\vartheta\tau_\gamma\big)'\Big(\frac{-1}{\vartheta\tau_\gamma}D^{\perp}_{T_\gamma}B_\gamma\Big) + \big(\vartheta\tau_\gamma\big)^2 B_\gamma \end{aligned}$$

olur ve buradan

$$D^{\perp}_{T_\gamma}D^{\perp}_{T_\gamma}B_\gamma - \frac{(\vartheta\tau_\gamma)'}{\vartheta\tau_\gamma}D^{\perp}_{T_\gamma}B_\gamma + \big(\vartheta\tau_\gamma\big)^2 B_\gamma = 0$$

elde edilir. □

Teorem 4.2.10 (α,γ) Bertrand eğri çifti olsun. γ partner eğrisinin normal konneksiyona göre diferensiyel denklemi α eğrisinin B binormal vektörü yönünde türevler cinsinden

1) α eğrisinin T teğet vektörüne bağlı denklemi:

$$\Big(\frac{1-cos\theta}{sin\theta}\kappa\Big)D^{\perp}_{B}D^{\perp}_{B}T - \Big(\frac{1-cos\theta}{sin\theta}\kappa'\Big)D^{\perp}_{B}T + \Big(\frac{1-cos\theta}{sin\theta}\kappa\Big)^3 T = 0,$$

2) α eğrisinin N asli normal vektörüne bağlı denklemi:

$$\Big(\frac{1-cos\theta}{sin\theta}\kappa\Big)D^{\perp}_{B}D^{\perp}_{B}N - \Big(\frac{1-cos\theta}{sin\theta}\kappa'\Big)D^{\perp}_{B}N + \Big(\frac{1-cos\theta}{sin\theta}\kappa\Big)^3 N = 0$$

şeklinde verilir.

İspat. (4.2.10) bağıntısından

$$D^{\perp}_{B}T = \big(\frac{1-cos\theta}{sin\theta}\kappa\big)N \quad \text{ve} \quad D^{\perp}_{B}N = -\big(\frac{1-cos\theta}{sin\theta}\kappa\big)T$$

yazılır. Bu ifadelerde T ve N vektörlerinin katsayısı

$$\frac{1-cos\theta}{sin\theta}\kappa = \alpha_1$$

ile gösterilsin. $D_B^{\perp}T = \alpha_1 N$ nin türevi alınırsa

$$\begin{aligned} D_B^{\perp}T = \alpha_1 N \implies D_B^{\perp}\big(D_B^{\perp}T\big) &= D_B^{\perp}\big(\alpha_1 N\big) \\ &= \alpha_1' N + \alpha_1 D_B^{\perp}N \\ &= \frac{\alpha_1'}{\alpha_1} D_B^{\perp}T - \alpha_1^2 T \end{aligned}$$

olur ve T ye bağlı denklem

$$\alpha_1 D_B^{\perp}D_B^{\perp}T - \alpha_1' D_B^{\perp}T + \alpha_1^3 T = 0$$

şeklinde elde edilir. Benzer şekilde

$$\begin{aligned} D_B^{\perp}N = -\alpha_1 T \implies D_B^{\perp}\big(D_B^{\perp}N\big) &= D_B^{\perp}\big(-\alpha_1 T\big) \\ &= -\alpha_1' T - \alpha_1 D_B^{\perp}T \\ &= \frac{\alpha_1'}{\alpha_1} D_B^{\perp}N - \alpha_1^2 N \end{aligned}$$

ifadesinden N ye bağlı denklem

$$\alpha_1 D_B^{\perp}D_B^{\perp}N - \alpha_1' D_B^{\perp}N + \alpha_1^3 N = 0$$

şeklinde elde edilir ve böylece ispat tamamlanır. □

Örnek 4.2.1 $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ verilsin. α eğrisinin Frenet vektörleri

$$T = \frac{1}{\sqrt{2}}(-sins, coss, 1), \quad N = (-coss, -sins, 0), \quad B = \frac{1}{\sqrt{2}}(sins, -coss, 1)$$

birinci ve ikinci eğrilikleri de sırasıyla, $\kappa = \frac{1}{\sqrt{2}}$, $\tau = \frac{1}{\sqrt{2}}$ dir. Frenet formülleri

$$D_T T = \frac{-1}{\sqrt{2}}(coss, sins, 0), \quad D_T N = (sins, -coss, 0), \quad D_T B = \frac{1}{\sqrt{2}}(coss, sins, 0)$$

dır. Buna göre α eğrisinin Bertrand partneri

$$\gamma(s) = \frac{1}{\sqrt{2}}(coss, sins, s) + \lambda(-coss, -sins, 0) \text{ dir.}$$

(4.2.1) bağıntısına göre γ eğrisinin diferensiyel denklemi

$$D_B^3 B + \Big(\frac{1 - cos\theta}{sin\theta}\Big)^2 D_B B = 0$$

olur. (4.2.10) bağıntısına göre γ eğrisinin diferensiyel denklemi

$$D_B^{\perp} D_B^{\perp} T + \Big(\frac{cos\theta - 1}{\sqrt{2}\, sin\theta}\Big)^2 T = 0$$

veya

$$D_B^{\perp} D_B^{\perp} N + \Big(\frac{cos\theta - 1}{\sqrt{2}\, sin\theta}\Big)^2 N = 0$$

şeklinde elde edilir.

4.2.4 Bertrand Partner Eğrisinin Darboux Vektörüne Göre Denklemleri

Ortalama eğrilik vektör alanı H_γ nin yerine Darboux vektörü W_γ konulursa, Bertrand partner eğrisi γ için harmoniklik koşulları ve diferensiyel denklemler W_γ vektörüne göre yeniden yorumlanabilir.

Teorem 4.2.11 (α, γ) bir Bertrand eğri çifti ve γ partner eğrisinin Darboux vektörü W_γ olsun. Konneksiyona göre γ eğrisinin diferensiyel denklemi

$$d_{\gamma 1} D^3_{T_\gamma} W_\gamma + d_{\gamma 2} D^2_{T_\gamma} W_\gamma + d_{\gamma 3} D_{T_\gamma} W_\gamma + d_{\gamma 4} W_\gamma = 0 \qquad (4.2.22)$$

bağıntısıyla verilir. Burada $d_{\gamma 1},\ d_{\gamma 2},\ d_{\gamma 3}$ ve $d_{\gamma 4}$

$$d_{\gamma 1} \;=\; \vartheta\Big(\kappa_\gamma (\tau_\gamma)' - (\kappa_\gamma)' \tau_\gamma\Big)^2,$$

$$d_{\gamma 2} = \Big(\vartheta(\kappa_\gamma)''\tau_\gamma - \vartheta\kappa_\gamma(\tau_\gamma)'' - \big(\vartheta\kappa_\gamma(\tau_\gamma)' - \vartheta(\kappa_\gamma)'\tau_\gamma\big)'\Big)\Big(\kappa_\gamma(\tau_\gamma)' - (\kappa_\gamma)'\tau_\gamma\Big),$$

$$d_{\gamma 3} = \Big((\kappa_\gamma)'''\tau_\gamma - \kappa_\gamma(\tau_\gamma)''' + \vartheta^2\big(\kappa_\gamma(\tau_\gamma)' - (\kappa_\gamma)'\tau_\gamma\big)\big((\kappa_\gamma)^2 + (\tau_\gamma)^2\big)\Big)$$

$$\Big(\vartheta\kappa_\gamma(\tau_\gamma)' - \vartheta(\kappa_\gamma)'\tau_\gamma\Big)$$

$$+\Big(\vartheta(\kappa_\gamma)''\tau_\gamma - \vartheta\kappa_\gamma(\tau_\gamma)'' - \big(\vartheta\kappa_\gamma(\tau_\gamma)' - \vartheta(\kappa_\gamma)'\tau_\gamma\big)'\Big)\Big((\kappa_\gamma)''\tau_\gamma - \kappa_\gamma(\tau_\gamma)''\Big),$$

$$d_{\gamma 4} = \Big((\kappa_\gamma)'(\tau_\gamma)''' - (\kappa_\gamma)'''(\tau_\gamma)' - \vartheta^2\big(\kappa_\gamma(\kappa_\gamma)' + \tau_\gamma(\tau_\gamma)'\big)\big(\kappa_\gamma(\tau_\gamma)' - (\kappa_\gamma)'\tau_\gamma\big)\Big)$$

$$\Big(\vartheta\kappa_\gamma(\tau_\gamma)' - \vartheta(\kappa_\gamma)'\tau_\gamma\Big)$$

$$+\Big(\vartheta(\kappa_\gamma)''\tau_\gamma - \vartheta\kappa_\gamma(\tau_\gamma)'' - \big(\vartheta\kappa_\gamma(\tau_\gamma)' - \vartheta(\kappa_\gamma)'\tau_\gamma\big)'\big((\kappa_\gamma)'(\tau_\gamma)'' - (\kappa_\gamma)''(\tau_\gamma)'\big)\Big)$$

şeklinde birer katsayıdır.

İspat. (3.1.5) bağıntısından $W_\gamma = \tau_\gamma T_\gamma + \kappa_\gamma B_\gamma$ yazılır. Bu vektörün T_γ yönünde birinci, ikinci ve üçüncü türevi alınırsa sırasıyla

$$\begin{aligned} D_{T_\gamma}W_\gamma &= \tau_\gamma' T_\gamma + \tau_\gamma D_{T_\gamma}T_\gamma + \kappa_\gamma' B_\gamma + \kappa_\gamma D_{T_\gamma}B_\gamma \\ &= \tau_\gamma' T_\gamma + \tau_\gamma\vartheta\kappa_\gamma N_\gamma + \kappa_\gamma' B_\gamma - \kappa_\gamma\vartheta\tau_\gamma N_\gamma \\ &= \tau_\gamma' T_\gamma + \kappa_\gamma' B_\gamma\,, \end{aligned}$$

$$
\begin{aligned}
D^2_{T_\gamma} W_\gamma &= D_{T_\gamma}\Big(\tau'_\gamma T_\gamma + \kappa'_\gamma B_\gamma\Big) \\
&= \tau''_\gamma T_\gamma + \tau'_\gamma D_{T_\gamma} T_\gamma + \kappa''_\gamma B_\gamma + \kappa'_\gamma D_{T_\gamma} B_\gamma \\
&= \tau''_\gamma T_\gamma + \big(\vartheta\kappa_\gamma \tau'_\gamma - \vartheta\kappa'_\gamma \tau_\gamma\big) N_\gamma + \kappa''_\gamma B_\gamma \,,
\end{aligned}
$$

$$
\begin{aligned}
D^3_{T_\gamma} W_\gamma &= D_{T_\gamma}\Big(\tau''_\gamma T_\gamma + \big(\vartheta\kappa_\gamma \tau'_\gamma - \vartheta\kappa'_\gamma \tau_\gamma\big) N_\gamma + \kappa''_\gamma B_\gamma\Big) \\
&= \tau'''_\gamma T_\gamma + \tau''_\gamma\big(\vartheta\kappa_\gamma N_\gamma\big) + \Big(\vartheta\kappa_\gamma \tau'_\gamma - \vartheta\kappa'_\gamma \tau_\gamma\Big)' N_\gamma \\
&\quad + \Big(\vartheta\kappa_\gamma \tau'_\gamma - \vartheta\kappa'_\gamma \tau_\gamma\Big)\Big(-\vartheta\kappa_\gamma T_\gamma + \vartheta\tau_\gamma B_\gamma\Big) + \kappa''_\gamma B_\gamma - \kappa''_\gamma \vartheta\tau_\gamma N_\gamma \\
&= \Big(\tau'''_\gamma - \vartheta^2\kappa_\gamma\big(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\big)\Big) T_\gamma + \Big(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + \big(\vartheta\kappa_\gamma \tau'_\gamma - \vartheta\kappa'_\gamma \tau_\gamma\big)'\Big) N_\gamma \\
&\quad + \Big(\vartheta^2\tau_\gamma\big(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\big) + \kappa'''_\gamma\Big) B_\gamma
\end{aligned}
$$

olur. W_γ ve $D_{T_\gamma} W_\gamma$ ifadelerinden T_γ ve B_γ vektörleri çekilirse

$$
\begin{aligned}
T_\gamma &= \frac{\kappa_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} D_{T_\gamma} W_\gamma - \frac{\kappa'_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} W_\gamma \,, \\
B_\gamma &= \frac{-\tau_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} D_{T_\gamma} W_\gamma + \frac{\tau'_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} W_\gamma
\end{aligned}
$$

bulunur. Bu vektörler $D^2_{T_\gamma} W_\gamma$ türevinde yerine yazılırsa N_γ vektörü

$$
\begin{aligned}
N_\gamma &= \frac{1}{\vartheta\big(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\big)} D^2_{T_\gamma} W_\gamma + \frac{\kappa''_\gamma\tau_\gamma - \kappa_\gamma\tau''_\gamma}{\vartheta\big(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\big)^2} D_{T_\gamma} W_\gamma \\
&\quad + \frac{\kappa'_\gamma\tau''_\gamma - \kappa''_\gamma\tau'_\gamma}{\vartheta\big(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\big)^2} W_\gamma
\end{aligned}
$$

şeklinde bulunur. T_γ, N_γ ve B_γ vektörleri $D^3_{T_\gamma}W_\gamma$ türevinde yerine yazılırsa

$$\begin{aligned}
D^3_{T_\gamma}W_\gamma &= \Big(\big(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\big)\frac{1}{\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma}\Big)D^2_{T_\gamma}W_\gamma \\
&+\Big(\big(\kappa_\gamma\tau'''_\gamma - \kappa'''_\gamma\tau_\gamma - \vartheta^2(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)(\kappa^2_\gamma + \tau^2_\gamma)\big)\frac{1}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} \\
&+\big(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\big)\frac{\kappa''_\gamma\tau_\gamma - \kappa_\gamma\tau''_\gamma}{\vartheta(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)^2}\Big)D_{T_\gamma}W_\gamma \\
&+\Big(\big(\kappa'''_\gamma\tau'_\gamma - \kappa'_\gamma\tau'''_\gamma + \vartheta^2(\kappa_\gamma\kappa'_\gamma + \tau_\gamma\tau'_\gamma)(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)\big)\frac{1}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} \\
&+\big(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\big)\frac{\kappa'_\gamma\tau''_\gamma - \kappa''_\gamma\tau'_\gamma}{\vartheta(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)^2}\Big)W_\gamma
\end{aligned}$$

elde edilir. Bu ifade $D^3_{T_\gamma}W_\gamma,\ D^2_{T_\gamma}W_\gamma,\ D_{T_\gamma}W_\gamma,\ W_\gamma$ nin lineer bileşimi olarak düzenlenir ve katsayılarına da sırasıyla $d_{\gamma1},\ d_{\gamma2},\ d_{\gamma3},\ d_{\gamma4}$ denilirse, ispat tamamlanır. □

Teorem 4.2.12 (α, γ) bir Bertrand eğri çifti ve α eğrisinin Darboux vektörü W olsun. Konneksiyona göre γ partner eğrisinin diferensiyel denklemi α eğrisinin Frenet aparatları türünden

$$\omega_1 D^3_B W + \omega_2 D^2_B W + \omega_3 D_B W + \omega_4 W = 0$$

bağıntısıyla verilir. Burada $\omega_1,\ \omega_2,\ \omega_3,\ \omega_4,\ \varrho_1,\ \varrho_2,\ \varrho_3,\ \varrho_4$ ve $d_1,\ d_2,\ d_3,\ d_4$

$$\omega_1 = d_1\frac{sin^2\theta\big(cos\theta + sin\theta\big)}{\tau(\lambda^2 + \mu^2)},$$

$$\omega_2 = d_1\frac{3sin^2\theta\big(cos\theta+sin\theta\big)}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)' + d_2\frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)} + \frac{\varrho_3 cos\theta}{(1-sin\theta)(\kappa\tau'-\kappa'\tau)}\,,$$

$$\omega_3 = \varrho_1 + \frac{\varrho_2\kappa-\varrho_4\tau}{\kappa\tau'-\kappa'\tau} + \frac{\varrho_3(\kappa\tau''-\kappa''\tau)cos\theta}{(sin\theta-1)(\kappa\tau'-\kappa'\tau)^2}\,,$$

$$\omega_4 = \frac{\varrho_4\tau'-\varrho_2\kappa'}{\kappa\tau'-\kappa'\tau} + \frac{\varrho_3(\kappa''\tau'-\kappa'\tau'')cos\theta}{(sin\theta-1)(\kappa\tau'-\kappa'\tau)^2}\,,$$

$$\varrho_1 = d_1\frac{sin\theta\big(2sin2\theta+2sin^2\theta+1\big)}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)'' + d_2\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)' + d_3\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)},$$

$$\begin{aligned}\varrho_2 = {} & d_1\frac{sin\theta\big(1+sin2\theta\big)\tau}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)''' + d_2\frac{(cos\theta+sin\theta)^2\tau}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)'' - d_3\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\big(\frac{\tau'}{\tau}\big)\\ & +d_1\big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\big)\Big(\big(\frac{\tau''}{\tau}\big)'sin\theta cos\theta + cos\theta\big(cos\theta-1\big)\big(\kappa\tau'-\kappa'\tau\big)\big(\frac{\kappa\tau+\kappa}{\tau}\big)\\ & -\big((\frac{\tau'}{\tau})^2\big)'sin2\theta + cos\theta\big(cos\theta+sin\theta\big)\big((\frac{1}{\tau})''\tau\big)' + cos\theta\big(cos\theta+sin\theta\big)\big(\frac{\tau''\tau-2(\tau')^2}{\tau^2}\big)'\\ & -\big(\frac{\kappa}{\tau}\big)cos^2\theta\big(\kappa\tau'-\kappa'\tau\big)\Big) + d_2\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau-2(\tau')^2\big) + \frac{d_4}{\lambda^2+\mu^2}\,,\end{aligned}$$

$$\begin{aligned}\varrho_3 = {} & d_1\big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\big)\Big(\big(\frac{\kappa\tau''}{\tau}\big)cos\theta sin\theta + cos\theta\big(1-cos\theta\big)\Big(\big(\kappa\tau'-\kappa'\tau\big)'\\ & +\big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau-2(\tau')^2\big) + \frac{2\kappa'\tau'-\kappa''\tau}{\tau}\Big) + \big(\frac{\kappa'\tau'}{\tau}\big)sin2\theta - \big(\kappa''\tau\big)cos\theta sin\theta\\ & -\kappa\big(\frac{\tau'}{\tau}\big)^2sin2\theta + \big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau-2(\tau')^2\big)cos^2\theta\end{aligned}$$

$$+cos\theta\big(cos\theta+sin\theta\big)\big(\frac{\kappa\tau'-\kappa'\tau}{\tau}\big)'-cos^2\theta\big(\frac{\kappa''\tau-2\kappa'\tau'}{\tau}\big)\Big)$$

$$+d_2\frac{cos\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau'-\kappa'\tau\big)\,,$$

$$\begin{aligned}\varrho_4 &= d_1\frac{sin\theta\big(1+sin2\theta\big)\kappa}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)'''+d_2\frac{(cos\theta+sin\theta)^2\kappa}{\lambda^2+\mu^2}(\frac{1}{\tau})''-d_3\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}(\frac{\kappa\tau'}{\tau^2})\\
&+\frac{\kappa d_4}{\tau(\lambda^2+\mu^2)}+d_1\big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\big)\Big(\big(\kappa\tau'-\kappa'\tau\big)\big(cos\theta+\tau(cos\theta-cos^2\theta)\big)\\
&+\kappa'''sin\theta cos\theta-\big(\frac{\kappa'\tau'}{\tau^2}\big)'sin2\theta+cos\theta\big(cos\theta+sin\theta\big)\Big(\big((\frac{1}{\tau})''\kappa\big)'+\big(\frac{\kappa''\tau-2\kappa'\tau'}{\tau^2}\big)'\Big)\Big)\\
&+d_2\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau-2\kappa'\tau'\big)\end{aligned}$$

ve $d_1,\ d_2,\ d_3,\ d_4$

$$d_1 = \tau\sqrt{\lambda^2+\mu^2}\Big(\frac{sin^2\theta}{\lambda^2\tau}\Big)^2\Bigg(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big(\frac{1}{\tau}\big)'-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Bigg)^2,$$

$$\begin{aligned}d_2 &= \Bigg(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)''-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)''\Big)\\
&-\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)'-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)'\Bigg)\\
&\Bigg(\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)'-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Bigg),\end{aligned}$$

$$
\begin{aligned}
d_3 \;=\; & \Big((\frac{sin^2\theta}{\lambda^2\tau}) \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})''' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})''' \Big) + \tau\big(\lambda^2+\mu^2\big)(\frac{sin^2\theta}{\lambda^2}) \\
& \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \Big) \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})^2 + (\frac{sin^2\theta}{\lambda^2\tau})^2 \Big) \Big) \\
& \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \Big) \Big) \\
& + \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})'' \Big) \\
& - \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta \big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \big) \Big)' \Big) \\
& \Big((\frac{sin^2\theta}{\lambda^2\tau}) \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})'' \Big) \Big),
\end{aligned}
$$

$$
\begin{aligned}
d_4 \;=\; & \Big((\frac{sin^2\theta}{\lambda^2}) \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{1}{\tau})''' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'''(\frac{1}{\tau})' \Big) \\
& - \big(\lambda^2+\mu^2\big) \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' + (\frac{sin^2\theta}{\lambda^2})^2(\frac{1}{\tau})' \Big) \\
& \Big(\frac{sin^2\theta}{\lambda^2} \Big) \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \Big) \Big)
\end{aligned}
$$

$$\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big)$$

$$+\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)''\Big)$$

$$-\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big)'$$

$$\frac{sin^2\theta}{\lambda^2}\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\big(\frac{1}{\tau}\big)'' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)''\big(\frac{1}{\tau}\big)'\Big)$$

şeklinde birer katsayıdır.

İspat. (4.2.22) denklemindeki W_γ vektörü (3.1.11) ve (3.1.12) bağıntılarına göre α eğrisinin Frenet aparatları türünden

$$W_\gamma = \frac{1}{\tau\big(\lambda^2+\mu^2\big)}W$$

şeklinde yazılır. Bu vektörün T_γ yönünde birinci türevi

$$\begin{aligned}
D_{T_\gamma}W_\gamma &= D_{T_\gamma}\Big(\frac{1}{\tau(\lambda^2+\mu^2)}W\Big) \\
&= D_{\big(cos\theta T + sin\theta B\big)}\Big(\frac{1}{\tau\big(\lambda^2+\mu^2\big)}W\Big) \\
&= cos\theta D_T\big(\frac{1}{\tau(\lambda^2+\mu^2)}W\big) + sin\theta D_B\big(\frac{1}{\tau(\lambda^2+\mu^2)}W\big)
\end{aligned}$$

olur. Burada önce $D_T\big(\frac{1}{\tau(\lambda^2+\mu^2)}W\big)$ ve $D_B\big(\frac{1}{\tau(\lambda^2+\mu^2)}W\big)$ vektörleri hesaplanırsa

$$
\begin{aligned}
D_T\Big(\frac{1}{\tau(\lambda^2+\mu^2)}W\Big) &= \big(\frac{1}{\tau(\lambda^2+\mu^2)}\big)'W + \big(\frac{1}{\tau(\lambda^2+\mu^2)}\big)D_T W \\
&= \big(\frac{1}{\tau}\big)'\frac{1}{\lambda^2+\mu^2}W + \frac{1}{\tau(\lambda^2+\mu^2)}\Big(D_T\big(\tau T+\kappa B\big)\Big) \\
&= \big(\frac{1}{\tau}\big)'\frac{1}{\lambda^2+\mu^2}W + \frac{1}{\tau(\lambda^2+\mu^2)}\Big(\tau' T+\kappa\tau N+\kappa' B-\kappa\tau N\Big) \\
&= \big(\frac{1}{\tau}\big)'\frac{1}{\lambda^2+\mu^2}W + \frac{1}{\tau(\lambda^2+\mu^2)}\Big(\tau' T+\kappa' B\Big)
\end{aligned}
$$

ve

$$
\begin{aligned}
D_B\Big(\frac{1}{\tau(\lambda^2+\mu^2)}W\Big) &= \big(\frac{1}{\tau(\lambda^2+\mu^2)}\big)'W + \big(\frac{1}{\tau(\lambda^2+\mu^2)}\big)D_B W \\
&= \big(\frac{1}{\tau}\big)'\frac{1}{\lambda^2+\mu^2}W + \frac{1}{\tau(\lambda^2+\mu^2)}\Big(D_B\big(\tau T+\kappa B\big)\Big) \\
&= \big(\frac{1}{\tau}\big)'\frac{1}{\lambda^2+\mu^2}W + \frac{1}{\tau(\lambda^2+\mu^2)}\Big(\tau' T+\kappa\tau\big(\frac{1-cos\theta}{sin\theta}\big)N \\
&\quad +\kappa' B-\kappa\tau\big(\frac{1-cos\theta}{sin\theta}\big)N\Big) \\
&= \big(\frac{1}{\tau}\big)'\frac{1}{\lambda^2+\mu^2}W + \frac{1}{\tau(\lambda^2+\mu^2)}\Big(\tau' T+\kappa' B\Big)
\end{aligned}
$$

bulunur. Yukarıdaki ifadelerden

$$D_T\Big(\frac{1}{\tau(\lambda^2+\mu^2)}W\Big)=D_B\Big(\frac{1}{\tau(\lambda^2+\mu^2)}W\Big)$$

eşitliği oluşur ve buradan

$$D_{T_\gamma}W_\gamma=\Big(\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)}\Big)D_BW+\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}(\frac{1}{\tau})'\Big)W$$

yazılır. W_γ vektörünün ikinci türevi

$$\begin{aligned}
D^2_{T_\gamma}W_\gamma &= D_{T_\gamma}\big(D_{T_\gamma}W_\gamma\big)\\
&= D_{(cos\theta T+sin\theta B)}\Big((\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)})D_BW+(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}(\frac{1}{\tau})')W\Big)\\
&= cos\theta D_T\Big((\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)})D_BW+(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}(\frac{1}{\tau})')W\Big)\\
&\quad +sin\theta D_B\Big((\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)})D_BW+(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}(\frac{1}{\tau})')W\Big)\\
&= cos\theta(\frac{cos\theta+sin\theta}{(\lambda^2+\mu^2)})(\frac{1}{\tau})'D_BW+cos\theta(\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)})D_T\big(D_BW\big)\\
&\quad +cos\theta(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2})(\frac{1}{\tau})''W+cos\theta(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2})(\frac{1}{\tau})'D_TW\\
&\quad +sin\theta(\frac{cos\theta+sin\theta}{(\lambda^2+\mu^2)})(\frac{1}{\tau})'D_BW+sin\theta(\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)})D_B\big(D_BW\big)
\end{aligned}$$

$$+sin\theta\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)(\frac{1}{\tau})''W+sin\theta\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)(\frac{1}{\tau})'D_BW.$$

Burada önce $D_T\big(D_BW\big)$ türevini hesaplamak gerekir. Bunun için D_BW vektörünün eşiti

$$\begin{aligned}
D_BW &= D_B\big(\tau T+\kappa B\big)\\
&= \tau'T+\tau D_BT+\kappa'B+\kappa D_BB\\
&= \tau'T+\kappa\tau\Big(\frac{1-cos\theta}{sin\theta}\Big)N+\kappa'B-\kappa\tau\Big(\frac{1-cos\theta}{sin\theta}\Big)N\\
&= \tau'T+\kappa'B
\end{aligned}$$

şeklinde hesaplanır ve $D^2_{T_\gamma}W_\gamma$ ikinci türevde yerine yazılırsa,

$$\begin{aligned}
D^2_{T_\gamma}W_\gamma &= cos\theta\Big(\frac{cos\theta+sin\theta}{(\lambda^2+\mu^2)}\Big)(\frac{1}{\tau})'D_BW+cos\theta\Big(\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)}\Big)D_T\big(\tau'T+\kappa'B\big)\\
&+cos\theta\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)(\frac{1}{\tau})''W+cos\theta\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)(\frac{1}{\tau})'D_TW\\
&+sin\theta\Big(\frac{cos\theta+sin\theta}{(\lambda^2+\mu^2)}\Big)(\frac{1}{\tau})'D_BW+sin\theta\Big(\frac{cos\theta+sin\theta}{\tau(\lambda^2+\mu^2)}\Big)D^2_BW\\
&+sin\theta\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)(\frac{1}{\tau})''W+sin\theta\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)(\frac{1}{\tau})'D_BW\\
&= \frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}D^2_BW+\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})'D_BW
\end{aligned}$$

$$+\frac{(cos\theta+sin\theta)^2}{\lambda^2+\mu^2}(\frac{1}{\tau})''W+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau-2(\tau')^2\big)T$$

$$+\frac{cos\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau'-\kappa'\tau\big)N+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau-2\kappa'\tau'\big)B \tag{4.2.23}$$

bulunur. Benzer şekilde W_γ vektörünün T_γ yönünde üçüncü türevi

$$\begin{aligned}
D^3_{T_\gamma}W_\gamma &= D_{T_\gamma}\big(D^2_{T_\gamma}W_\gamma\big)\\
&= D_{(cos\theta T+sin\theta B)}\Big(\frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}D^2_BW+\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})'D_BW\\
&\quad+\frac{(cos\theta+sin\theta)^2}{\lambda^2+\mu^2}(\frac{1}{\tau})''W+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau-2(\tau')^2\big)T\\
&\quad+\frac{cos\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau'-\kappa'\tau\big)N+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau-2\kappa'\tau'\big)B\Big)\\
&= cos\theta D_T\Big(\frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}D^2_BW+\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})'D_BW\\
&\quad+\frac{(cos\theta+sin\theta)^2}{\lambda^2+\mu^2}(\frac{1}{\tau})''W+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau-2(\tau')^2\big)T\\
&\quad+\frac{cos\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau'-\kappa'\tau\big)N+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau-2\kappa'\tau'\big)B\Big)
\end{aligned}$$

$$+sin\theta D_B\Big(\frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}D_B^2W+\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})'D_BW$$

$$+\frac{(cos\theta+sin\theta)^2}{\lambda^2+\mu^2}(\frac{1}{\tau})''W+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau-2(\tau')^2\big)T$$

$$+\frac{cos\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau'-\kappa'\tau\big)N+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau-2\kappa'\tau'\big)B\Big).$$

Burada

$$D_T\Big(\frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}D_B^2W\Big)\quad \text{ve}\quad D_T\Big(\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})'D_BW\Big)$$

ifadelerini hesaplayabilmek için önce D_BW ve D_B^2W türevleri

$$\begin{aligned}
D_BW &= \tau'T+\kappa'B\,,\\
D_B^2W &= D_B\big(D_BW\big)\\
&= D_B\big(\tau'T+\kappa'B\big)\\
&= \tau''T+\tau'D_BT+\kappa''B+\kappa'D_BB\\
&= \tau''T+\big(\frac{1-cos\theta}{sin\theta}\big)\big(\kappa\tau'-\kappa'\tau\big)N+\kappa''B
\end{aligned}$$

şeklinde hesaplanır ve $D_{T_\gamma}^3W_\gamma$ üçüncü türevde yerine yazılırsa,

$$
\begin{aligned}
D^3_{T_\gamma}W_\gamma &= cos\theta D_T\Big(\frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\tau''T+\big(\frac{1-cos\theta}{sin\theta}\big)\big(\kappa\tau'-\kappa'\tau\big)N+\kappa''B\big) \\
&+\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})'\big(\tau'T+\kappa'B\big) \\
&+\frac{(cos\theta+sin\theta)^2}{\lambda^2+\mu^2}(\frac{1}{\tau})''W+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau-2(\tau')^2\big)T \\
&+\frac{cos\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau'-\kappa'\tau\big)N+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau-2\kappa'\tau'\big)B\Big) \\
&+sin\theta D_B\Big(\frac{sin\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}D^2_BW+\frac{2sin\theta(cos\theta+sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})'D_BW \\
&+\frac{(cos\theta+sin\theta)^2}{\lambda^2+\mu^2}(\frac{1}{\tau})''W+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau-2(\tau')^2\big)T \\
&+\frac{cos\theta(cos\theta+sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau'-\kappa'\tau\big)N+\frac{cos\theta(cos\theta+sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau-2\kappa'\tau'\big)B\Big) \\
&= \frac{sin^2\theta\big(cos\theta+sin\theta\big)}{\tau(\lambda^2+\mu^2)}D^3_BW+\frac{3sin^2\theta\big(cos\theta+sin\theta\big)}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)'D^2_BW \\
&+\frac{sin\theta\big(2sin2\theta+2sin^2\theta+1\big)}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)''D_BW+\frac{sin\theta\big(1+sin2\theta\big)}{\lambda^2+\mu^2}\big(\frac{1}{\tau}\big)'''W
\end{aligned}
$$

$$+\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)\Big(\big(\frac{\tau''}{\tau}\big)'sin\theta cos\theta+cos\theta\big(cos\theta-1\big)\big(\kappa\tau'-\kappa'\tau\big)\big(\frac{\kappa\tau+\kappa}{\tau}\big)$$

$$-\big((\frac{\tau'}{\tau})^2\big)'sin2\theta+cos\theta\big(cos\theta+sin\theta\big)\big((\frac{1}{\tau})''\tau\big)'$$

$$+cos\theta\big(cos\theta+sin\theta\big)\big(\frac{\tau''\tau-2(\tau')^2}{\tau^2}\big)'-\big(\frac{\kappa}{\tau}\big)cos^2\theta\big(\kappa\tau'-\kappa'\tau\big)\Big)T$$

$$+\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)\Big(\big(\frac{\kappa\tau''}{\tau}\big)cos\theta sin\theta+cos\theta\big(1-cos\theta\big)\Big(\big(\kappa\tau'-\kappa'\tau\big)'$$

$$+\big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau-2(\tau')^2\big)+\frac{2\kappa'\tau'-\kappa''\tau}{\tau}\Big)+\big(\frac{\kappa'\tau'}{\tau}\big)sin2\theta-\big(\kappa''\tau\big)cos\theta sin\theta$$

$$-\kappa\big(\frac{\tau'}{\tau}\big)^2sin2\theta+\big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau-2(\tau')^2\big)cos^2\theta$$

$$+cos\theta\big(cos\theta+sin\theta\big)\big(\frac{\kappa\tau'-\kappa'\tau}{\tau}\big)'-cos^2\theta\big(\frac{\kappa''\tau-2\kappa'\tau'}{\tau}\big)\Big)N$$

$$+\Big(\frac{cos\theta+sin\theta}{\lambda^2+\mu^2}\Big)\Big(\big(\kappa\tau'-\kappa'\tau\big)\big(cos\theta+\tau(cos\theta-cos^2\theta)\big)+\kappa'''sin\theta cos\theta$$

$$-\big(\frac{\kappa'\tau'}{\tau^2}\big)'sin2\theta+cos\theta\big(cos\theta+sin\theta\big)\Big(\big((\frac{1}{\tau})''\kappa\big)'+\big(\frac{\kappa''\tau-2\kappa'\tau'}{\tau^2}\big)'\Big)\Big)B$$

olur. $D_{T_\gamma}W_\gamma$, $D^2_{T_\gamma}W_\gamma$, $D^3_{T_\gamma}W_\gamma$ ifadelerindeki T, N, B vektörleri de (4.2.1) bağıntısına göre W türünden hesaplanırsa

$$T = \Big(\frac{\kappa}{\kappa\tau' - \kappa'\tau}\Big) D_B W - \Big(\frac{\kappa'}{\kappa\tau' - \kappa'\tau}\Big) W,$$

$$N = \Big(\frac{cos\theta}{(1-sin\theta)(\kappa\tau' - \kappa'\tau)}\Big) D_B^2 W + \Big(\frac{cos\theta(\kappa\tau'' - \kappa''\tau)}{(sin\theta - 1)(\kappa\tau' - \kappa'\tau)^2}\Big) D_B W$$

$$+\Big(\frac{cos\theta(\kappa''\tau' - \kappa'\tau'')}{(sin\theta - 1)(\kappa\tau' - \kappa'\tau)^2}\Big) W,$$

$$B = \Big(\frac{-\tau}{\kappa\tau' - \kappa'\tau}\Big) D_B W + \Big(\frac{\tau'}{\kappa\tau' - \kappa'\tau}\Big) W$$

şeklinde bulunur. Son olarak (4.2.22) denkleminin katsayıları $d_{\gamma 1}$, $d_{\gamma 2}$, $d_{\gamma 3}$, $d_{\gamma 4}$ (3.1.11) ve (3.1.12) bağıntılarına göre α eğrisinin Frenet aparatları türünden yazılır ve bunlar sırasıyla $d_1,\ d_2,\ d_3,\ d_4$ ile gösterilirse,

$$d_1 = \tau\sqrt{\lambda^2+\mu^2}\Big(\frac{sin^2\theta}{\lambda^2\tau}\Big)^2\Bigg(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu}(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'\Bigg)^2,$$

$$d_2 = \Big(\tau\sqrt{\lambda^2+\mu^2}(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})''(\frac{sin^2\theta}{\lambda^2\tau}) - \tau\sqrt{\lambda^2+\mu^2}(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})''$$

$$-\Big(\tau\sqrt{\lambda^2+\mu^2}(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2\tau})(\frac{1}{\tau})' - \tau\sqrt{\lambda^2+\mu^2}(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{sin^2\theta}{\lambda^2\tau})\Big)'\Big)$$

$$\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{sin^2\theta}{\lambda^2\tau})\Big)$$

$$= \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)''\Big)$$

$$-\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\Big)'\Big)$$

$$\Big(\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big),$$

$$d_3 = \Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'''\big(\frac{sin^2\theta}{\lambda^2\tau}\big) - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)\big(\frac{sin^2\theta}{\lambda^2}\big)\big(\frac{1}{\tau}\big)'''$$

$$+\big(\tau\sqrt{\lambda^2+\mu^2}\big)^2\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)\big(\frac{sin^2\theta}{\lambda^2}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big)$$

$$\Big(\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)^2 + \big(\frac{sin^2\theta}{\lambda^2\tau}\big)^2\Big)\Big(\tau\sqrt{\lambda^2+\mu^2}\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)\big(\frac{sin^2\theta}{\lambda^2}\big)\big(\frac{1}{\tau}\big)'$$

$$-\tau\sqrt{\lambda^2+\mu^2}\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big)$$

$$+\Big(\tau\sqrt{\lambda^2+\mu^2}\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)''\big(\frac{sin^2\theta}{\lambda^2\tau}\big) - \tau\sqrt{\lambda^2+\mu^2}\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)\big(\frac{sin^2\theta}{\lambda^2}\big)\big(\frac{1}{\tau}\big)''$$

$$-\Big(\tau\sqrt{\lambda^2+\mu^2}\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)\big(\frac{sin^2\theta}{\lambda^2}\big)\big(\frac{1}{\tau}\big)' - \tau\sqrt{\lambda^2+\mu^2}\big(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau}\big)'\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big)'\Big)$$

$$\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})''(\frac{sin^2\theta}{\lambda^2\tau}) - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})''\Big)$$

$$= \Big((\frac{sin^2\theta}{\lambda^2\tau})\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})''' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})'''\Big) + \tau\big(\lambda^2 + \mu^2\big)(\frac{sin^2\theta}{\lambda^2})$$

$$\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'\Big)\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})^2 + (\frac{sin^2\theta}{\lambda^2\tau})^2\Big)\Big)$$

$$\Big(\frac{\sqrt{\lambda^2 + \mu^2}}{\lambda^2} sin^2\theta \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'\Big)\Big)$$

$$+\Big(\frac{\sqrt{\lambda^2 + \mu^2}}{\lambda^2} sin^2\theta \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})''\Big)$$

$$-\Big(\frac{\sqrt{\lambda^2 + \mu^2}}{\lambda^2} sin^2\theta ((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})')\Big)'\Big)$$

$$\Big((\frac{sin^2\theta}{\lambda^2\tau})\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})''\Big)\Big),$$

$$d_4 = \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \frac{sin^2\theta}{\lambda^2}(\frac{1}{\tau})''' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'''(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})'$$

$$-\tau^2\big(\lambda^2 + \mu^2\big)\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' + (\frac{sin^2\theta}{\lambda^2\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})'\Big)$$

$$\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{sin^2\theta}{\lambda^2\tau})\Big)\Big)$$

$$\Big(\tau\sqrt{\lambda^2+\mu^2}\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{sin^2\theta}{\lambda^2\tau})\Big)\Big)$$

$$+\Big(\tau\sqrt{\lambda^2+\mu^2}\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})''(\frac{sin^2\theta}{\lambda^2\tau}) - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})''\Big)$$

$$-\Big(\tau\sqrt{\lambda^2+\mu^2}\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{sin^2\theta}{\lambda^2\tau})\Big)\Big)'$$

$$\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})'' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})''(\frac{sin^2\theta}{\lambda^2})(\frac{1}{\tau})'\Big)\Big)$$

$$= \quad \Big((\frac{sin^2\theta}{\lambda^2})\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{1}{\tau})''' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'''(\frac{1}{\tau})'\Big)$$

$$-(\lambda^2+\mu^2)\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' + (\frac{sin^2\theta}{\lambda^2})^2(\frac{1}{\tau})'\Big)$$

$$\Big(\frac{sin^2\theta}{\lambda^2}\Big)\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'\Big)\Big)$$

$$\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)'-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big)$$

$$+\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)''-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)''\Big)$$

$$-\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)'-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big)'$$

$$\frac{sin^2\theta}{\lambda^2}\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\big(\frac{1}{\tau}\big)''-\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)''\big(\frac{1}{\tau}\big)'\Big)\Big)$$

şeklinde bulunur. Bu katsayılar (4.2.22) bağıntısında yerine yazılırsa, γ partner eğrisinin konneksiyona göre diferensiyel denklemi α eğrisinin Frenet aparatları cinsinden elde edilmiş olur. $\square$

Teorem 4.2.13 (α,γ) bir Bertrand eğri çifti ve γ partner eğrisinin birim Darboux vektörü C_γ olsun. Konneksiyona göre γ eğrisinin diferensiyel denklemi

$$D^3_{T_\gamma}C_\gamma+\eta_{\gamma 1}D^2_{T_\gamma}C_\gamma+\eta_{\gamma 2}D_{T_\gamma}C_\gamma+\eta_{\gamma 3}C_\gamma=0 \qquad (4.2.24)$$

bağıntısıyla verilir. Burada $\eta_{\gamma 1}$, $\eta_{\gamma 2}$ ve $\eta_{\gamma 3}$

$$\begin{aligned}
\eta_{\gamma 1} &= \Big(\frac{-\theta''_\gamma}{\theta'_\gamma}-\frac{\big(\theta'_\gamma\vartheta\parallel W_\gamma\parallel\big)'}{\vartheta\parallel W_\gamma\parallel\theta'_\gamma}\Big),\\
\eta_{\gamma 2} &= \big(\vartheta\parallel W_\gamma\parallel\big)^2+\big(\theta'_\gamma\big)^2-\big(\frac{\theta''_\gamma}{\theta'_\gamma}\big)'+\frac{\big(\theta'_\gamma\vartheta\parallel W_\gamma\parallel\big)'}{\vartheta\parallel W_\gamma\parallel\big(\theta'_\gamma\big)^2}\theta''_\gamma,\\
\eta_{\gamma 3} &= \Big(\big(\theta'_\gamma\big)^2\Big)'-\frac{\big(\theta'_\gamma\vartheta\parallel W_\gamma\parallel\big)'}{\vartheta\parallel W_\gamma\parallel}\theta'_\gamma
\end{aligned}$$

şeklinde birer katsayıdır.

İspat. (3.1.6) bağıntısından C_γ birim Darboux vektörü

$$C_\gamma = sin\theta_\gamma T_\gamma + cos\theta_\gamma B_\gamma$$

dir. Bu vektörün T_γ yönünde birinci türevi

$$\begin{aligned} D_{T_\gamma} C_\gamma &= D_{T_\gamma}\big(sin\theta_\gamma T_\gamma + cos\theta_\gamma B_\gamma\big) \\ &= \theta'_\gamma cos\theta_\gamma T_\gamma + sin\theta_\gamma \vartheta \kappa_\gamma N_\gamma - \theta'_\gamma sin\theta_\gamma B_\gamma - cos\theta_\gamma \vartheta \tau_\gamma N_\gamma \\ &= \theta'_\gamma\big(cos\theta_\gamma T_\gamma - sin\theta_\gamma B_\gamma\big) \end{aligned}$$

olur. C_γ ve $D_{T_\gamma} C_\gamma$ ifadelerinden T_γ ve B_γ vektörleri çekilirse

$$T_\gamma = sin\theta_\gamma C_\gamma + \frac{cos\theta_\gamma}{\theta'_\gamma} D_{T_\gamma} C_\gamma \quad \text{ve} \quad B_\gamma = cos\theta_\gamma C_\gamma - \frac{sin\theta_\gamma}{\theta'_\gamma} D_{T_\gamma} C_\gamma$$

bulunur. Bu vektörler C_γ nin ikinci türevinde yerine yazılırsa,

$$\begin{aligned} D^2_{T_\gamma} C_\gamma &= D_{T_\gamma}\Big(\theta'_\gamma\big(cos\theta_\gamma T_\gamma - sin\theta_\gamma B_\gamma\big)\Big) \\ &= \theta''_\gamma\Big(cos\theta_\gamma T_\gamma - sin\theta_\gamma B_\gamma\Big) - \big(\theta'_\gamma\big)^2\Big(sin\theta_\gamma T_\gamma + cos\theta_\gamma B_\gamma\Big) \\ &\quad + \theta'_\gamma \vartheta\Big(cos\theta_\gamma \kappa_\gamma + sin\theta_\gamma \tau_\gamma\Big) N_\gamma \\ &= \theta''_\gamma cos\theta_\gamma\Big(sin\theta_\gamma C_\gamma + \frac{cos\theta_\gamma}{\theta'_\gamma} D_{T_\gamma} C_\gamma\Big) - \theta''_\gamma sin\theta_\gamma\Big(cos\theta_\gamma C_\gamma - \frac{sin\theta_\gamma}{\theta'_\gamma} D_{T_\gamma} C_\gamma\Big) \\ &\quad + \theta'_\gamma \vartheta \parallel W_\gamma \parallel N_\gamma - \big(\theta'_\gamma\big)^2 \end{aligned}$$

$$= \frac{\theta''_\gamma}{\theta'_\gamma} D_{T_\gamma} C_\gamma - \big(\theta'_\gamma\big)^2 C_\gamma + \theta'_\gamma \vartheta \parallel W_\gamma \parallel N_\gamma$$

bulunur. Bu eşitlikten N_γ vektörü çekilirse

$$N_\gamma = \frac{1}{\vartheta \big(\theta'_\gamma\big)^2 \parallel W_\gamma \parallel} \Big(\theta'_\gamma D^2_{T_\gamma} C_\gamma - \theta''_\gamma D_{T_\gamma} C_\gamma + \big(\theta'_\gamma\big)^3 C_\gamma \Big)$$

olur. C_γ vektörünün T_γ yönünde üçüncü türevi

$$\begin{aligned}
D^3_{T_\gamma} C_\gamma &= D_{T_\gamma} \Big(\frac{\theta''_\gamma}{\theta'_\gamma} D_{T_\gamma} C_\gamma - \big(\theta'_\gamma\big)^2 C_\gamma + \theta'_\gamma \vartheta \parallel W_\gamma \parallel N_\gamma \Big) \\
&= \big(\frac{\theta''_\gamma}{\theta'_\gamma}\big)' D_{T_\gamma} C_\gamma + \frac{\theta''_\gamma}{\theta'_\gamma} D^2_{T_\gamma} C_\gamma - \Big(\big(\theta'_\gamma\big)^2 \Big)' C_\gamma - \big(\theta'_\gamma\big)^2 D_{T_\gamma} C_\gamma \\
&\quad + \vartheta^2 \theta'_\gamma \parallel W_\gamma \parallel \Big(- \kappa_\gamma T_\gamma + \tau_\gamma B_\gamma \Big) \\
&\quad + \big(\vartheta \theta'_\gamma \parallel W_\gamma \parallel \big)' \frac{1}{\vartheta \big(\theta'_\gamma\big)^2 \parallel W_\gamma \parallel} \Big(\theta'_\gamma D^2_{T_\gamma} C_\gamma - \theta''_\gamma D_{T_\gamma} C_\gamma + \big(\theta'_\gamma\big)^3 C_\gamma \Big)
\end{aligned}$$

$$\begin{aligned}
&= \Big(\frac{\theta''_\gamma}{\theta'_\gamma} + \frac{\big(\theta'_\gamma \vartheta \parallel W_\gamma \parallel\big)'}{\vartheta \parallel W_\gamma \parallel \theta'_\gamma} \Big) D^2_{T_\gamma} C_\gamma \\
&\quad + \Big(\big(\frac{\theta''_\gamma}{\theta'_\gamma}\big)' - \big(\vartheta \parallel W_\gamma \parallel\big)^2 - \big(\theta'_\gamma\big)^2 - \frac{\big(\theta'_\gamma \vartheta \parallel W_\gamma \parallel\big)'}{\vartheta \parallel W_\gamma \parallel \big(\theta'_\gamma\big)^2} \theta''_\gamma \Big) D_{T_\gamma} C_\gamma \\
&\quad + \Big(\frac{\big(\theta'_\gamma \vartheta \parallel W_\gamma \parallel\big)'}{\vartheta \parallel W_\gamma \parallel} \theta'_\gamma - \big((\theta'_\gamma)^2\big)' \Big) C_\gamma
\end{aligned}$$

şeklinde bulunur. Bu ifade $D^3_{T_\gamma} C_\gamma$, $D^2_{T_\gamma} C_\gamma$, $D_{T_\gamma} C_\gamma$ ve C_γ nin lineer bileşimi olarak düzenlenir ve katsayılarına da $\eta_{\gamma 1}$, $\eta_{\gamma 2}$ ve $\eta_{\gamma 3}$ denilirse, istenilen diferensiyel denklem elde edilir. □

Teorem 4.2.14 (α, γ) bir Bertrand eğri çifti ve α eğrisinin birim Darboux vektörü C olsun. Konneksiyona göre γ partner eğrisinin diferensiyel denklemi α eğrisinin Frenet aparatları türünden

$$\varepsilon_1 D_B^3 C + \varepsilon_2 D_B^2 C + \varepsilon_3 D_B C + \varepsilon_4 C = 0 \tag{4.2.25}$$

bağıntısıyla verilir. Burada $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4,\ \xi_1,\ \xi_2,\ \xi_3$ ve $\eta_1,\ \eta_2,\ \eta_3$

$$\begin{aligned}
\varepsilon_1 &= sin^2\theta\big(sin\theta + cos\theta\big), \\
\varepsilon_2 &= \eta_1 sin\theta\big(sin\theta + cos\theta\big) + \frac{\xi_2 sin\theta\sqrt{\kappa^2+\tau^2}}{(\kappa\tau' - \kappa'\tau)(1-cos\theta)}, \\
\varepsilon_3 &= \eta_2\big(sin\theta + cos\theta\big) + \big(\xi_1\kappa - \xi_3\tau\big)\frac{\sqrt{\kappa^2+\tau^2}}{\kappa\tau' - \kappa'\tau} \\
&\quad + \frac{\xi_2 sin\theta\big(\kappa^2+\tau^2\big)}{\big(\kappa\tau' - \kappa'\tau\big)^2\big(cos\theta - 1\big)}\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''\Big), \\
\varepsilon_4 &= \frac{\kappa^2+\tau^2}{\kappa\tau' - \kappa'\tau}\Big(\xi_3\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \xi_1\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big) \\
&\quad + \frac{\xi_2 sin\theta\big(\kappa^2+\tau^2\big)^{3/2}}{\big(\kappa\tau' - \kappa'\tau\big)^2\big(cos\theta - 1\big)}\left(\frac{\big(\kappa/\sqrt{\kappa^2+\tau^2}\big)'}{\big(\tau/\sqrt{\kappa^2+\tau^2}\big)'}\right)'\Big(\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)^2,
\end{aligned}$$

$$\begin{aligned}
\xi_1 &= \frac{\eta_3\tau}{\sqrt{\kappa^2+\tau^2}} + \Big(sin\theta + cos\theta\Big)\bigg(\Big(cos^2\theta + sin2\theta\Big)\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''' \\
&\quad + \Big(cos^2\theta - 2cos\theta\Big)\kappa\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big) + \eta_1 cos\theta\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''\bigg),
\end{aligned}$$

$$\begin{aligned}
\xi_2 &= \Big(sin\theta + cos\theta\Big)\bigg(cos\theta\Big(1 + sin\theta\Big)\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'' \\
&\quad + \Big(\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)\Big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''\Big) \\
&\quad + \eta_1 cos\theta\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)\bigg),
\end{aligned}$$

$$\begin{aligned}
\xi_3 &= \Big(sin\theta + cos\theta\Big)\bigg(\Big(\Big(2cos\theta - cos^2\theta\Big)\tau\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big) \\
&\quad + \Big(sin2\theta + cos^2\theta\Big)\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'''\Big) + \eta_1 cos\theta\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''\bigg) + \frac{\eta_3\kappa}{\sqrt{\kappa^2+\tau^2}},
\end{aligned}$$

$$\eta_1 = -\frac{\Big(arctan\frac{1}{\mu\kappa-\lambda\tau}\Big)''}{\Big(arctan\frac{1}{\mu\kappa-\lambda\tau}\Big)'} - \frac{\Big(\sqrt{\kappa^2+\tau^2}\Big(arctan\frac{1}{\mu\kappa-\lambda\tau}\Big)'\Big)'}{\sqrt{\kappa^2+\tau^2}\Big(arctan\frac{1}{\mu\kappa-\lambda\tau}\Big)'},$$

$$\eta_2 = \kappa^2+\tau^2+\Big(\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\Big)^2-\Big(\frac{\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)''}{\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'}\Big)'$$

$$+\frac{\Big(\sqrt{\kappa^2+\tau^2}\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\Big)'}{\sqrt{\kappa^2+\tau^2}\Big(\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\Big)^2}\Big(arctan\frac{1}{\mu\kappa-\lambda\tau}\Big)'',$$

$$\eta_3 = \Big(\big(\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\big)^2\Big)'-\frac{\Big(\sqrt{\kappa^2+\tau^2}\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\Big)'}{\sqrt{\kappa^2+\tau^2}}\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'$$

şeklinde birer katsayıdır.

İspat. (4.2.24) denklemindeki C_γ vektörü, (3.1.11) ve (3.1.12) bağıntılarına göre α eğrisinin Frenet aparatları türünden

$$C_\gamma = C$$

şeklinde yazılır. Bu vektörün T_γ yönünde birinci türevi

$$\begin{aligned} D_{T_\gamma}C_\gamma &= D_{T_\gamma}C \\ &= D_{(cos\theta T+sin\theta B)}C \\ &= cos\theta D_T C+sin\theta D_B C. \end{aligned}$$

Burada önce $D_T C$ ve $D_B C$ türevleri hesaplanırsa,

$$D_T C = D_T\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}T+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B\big)$$

$$= \Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}D_T T + \Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'B + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}D_T B$$

$$= \Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'T + \Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'B$$

ve

$$\begin{aligned} D_B C &= D_B\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B\Big) \\ &= \Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}D_B T + \Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'B + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}D_B B \\ &= \Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'T + \Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'B \end{aligned}$$

bulunur. Bu iki eşitlikten $D_T C = D_B C$ yazılır. Böylece

$$D_{T_\gamma} C_\gamma = \big(sin\theta + cos\theta\big) D_B C$$

olur. C_γ vektörünün T_γ yönünde ikinci türevi

$$\begin{aligned} D^2_{T_\gamma} C_\gamma &= D_{T_\gamma}\big(D_{T_\gamma} C_\gamma\big) \\ &= D_{(cos\theta T + sin\theta B)}\Big(\big(sin\theta + cos\theta\big) D_B C\Big) \\ &= cos\theta D_T\Big(\big(sin\theta + cos\theta\big) D_B C\Big) + sin\theta D_B\Big(\big(sin\theta + cos\theta\big) D_B C\Big) \\ &= cos\theta\big(sin\theta + cos\theta\big) D_T\big(D_B C\big) + sin\theta\big(sin\theta + cos\theta\big) D_B\big(D_B C\big). \end{aligned}$$

Burada $D_T\big(D_B C\big)$ vektörü

$$D_T\big(D_B C\big) = D_T\Big(\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'T + \Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'B\Big)$$

şeklinde hesaplanarak $D^2_{T_\gamma}C_\gamma$ ikinci türevde yerine yazılırsa,

$$\begin{aligned}
D^2_{T_\gamma}C_\gamma &= \Big(sin\theta + cos\theta\Big)\Bigg(cos\theta D_T\Big(\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'T + \big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'B\Big) + sin\theta D^2_B C\Bigg)\\
&= \Big(sin\theta\big(sin\theta + cos\theta\big)\Big)D^2_B C + \Big(cos\theta\big(sin\theta + cos\theta\big)\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''\Big)T\\
&\quad +\Big(cos\theta\big(sin\theta + cos\theta\big)\big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\big)\Big)N\\
&\quad +\Big(cos\theta\big(sin\theta + cos\theta\big)\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''\Big)B \qquad (4.2.26)
\end{aligned}$$

olur. Benzer şekilde C_γ vektörünün T_γ yönünde üçüncü türevi

$$\begin{aligned}
D^3_{T_\gamma}C_\gamma &= D_{T_\gamma}\big(D^2_{T_\gamma}C_\gamma\big)\\
&= D_{(cos\theta T + sin\theta B)}\Big(sin\theta\big(sin\theta + cos\theta\big)D^2_B C + cos\theta\big(sin\theta + cos\theta\big)\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''T\\
&\quad +cos\theta\big(sin\theta + cos\theta\big)\big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\big)N\\
&\quad +cos\theta\big(sin\theta + cos\theta\big)\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''B\Big)\\
&= cos\theta D_T\Big(sin\theta\big(sin\theta + cos\theta\big)D^2_B C + cos\theta\big(sin\theta + cos\theta\big)\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''T\\
&\quad +cos\theta\big(sin\theta + cos\theta\big)\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' - \tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)N\\
&\quad +cos\theta\big(sin\theta + cos\theta\big)\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''B\Big)
\end{aligned}$$

$$+sin\theta D_B\Big(sin\theta\big(sin\theta+cos\theta\big)D_B^2C+cos\theta\big(sin\theta+cos\theta\big)\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''T$$

$$+cos\theta\big(sin\theta+cos\theta\big)\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'-\tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)N$$

$$+cos\theta\big(sin\theta+cos\theta\big)\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''B\Big).$$

Burada önce

$$cos\theta D_T\Big(sin\theta\big(sin\theta+cos\theta\big)D_B^2C\Big)=cos\theta sin\theta\big(sin\theta+cos\theta\big)D_T\big(D_B^2C\big)$$

eşitliğindeki $D_T\big(D_B^2C\big)$ ifadesi

$$\begin{aligned} D_T\big(D_B^2C\big) &= D_T\Big(D_B\Big(\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'T+\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'B\Big)\Big) \\ &= D_T\Big(\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''T+\big(\frac{1-cos\theta}{sin\theta}\big)\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)' \\ &\quad -\tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)N+\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''B\Big) \end{aligned}$$

şeklinde hesaplanarak $D^3_{T_\gamma}C_\gamma$ üçüncü türevde yerine yazılırsa,

$$D^3_{T_\gamma}C_\gamma = sin\theta cos\theta\big(sin\theta+cos\theta\big)D_T\Big(\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''T\Big)+sin\theta cos\theta\big(sin\theta+cos\theta\big)$$

$$D_T\Big(\big(\frac{1-cos\theta}{sin\theta}\big)\big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'-\tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\big)N\Big)+sin\theta cos\theta\big(sin\theta+cos\theta\big)$$

$$D_T\Big(\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''B\Big)+cos^2\theta\big(sin\theta+cos\theta\big)D_T\Big(\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''T\Big)$$

$$+cos^2\theta(sin\theta+cos\theta)D_T\Big((\kappa(\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'-\tau(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})')N\Big)$$

$$+cos^2\theta\big(sin\theta+cos\theta\big)D_T\Big((\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})''B\Big)+sin^2\theta\big(sin\theta+cos\theta\big)D_B^3C$$

$$+sin\theta cos\theta\big(sin\theta+cos\theta\big)D_B\Big((\frac{\tau}{\sqrt{\kappa^2+\tau^2}})''T\Big)$$

$$+sin\theta cos\theta\big(sin\theta+cos\theta\big)D_B\Big((\kappa(\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'-\tau(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})')N\Big)$$

$$+sin\theta cos\theta\big(sin\theta+cos\theta\big)D_B\Big((\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})''B\Big)$$

$$=\quad\Big(sin^2\theta\big(sin\theta+cos\theta\big)\Big)D_B^3C+\Big(sin\theta+cos\theta\Big)\Big(\big(cos^2\theta+sin2\theta\big)(\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'''$$

$$+\big(cos^2\theta-2cos\theta\big)\kappa\big(\kappa(\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'-\tau(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})'\big)\Big)T$$

$$+\Big(cos\theta\big(1+sin\theta\big)\big(sin\theta+cos\theta\big)\Big)\Big(\kappa(\frac{\tau}{\sqrt{\kappa^2+\tau^2}})''$$

$$+\Big(\big(\kappa(\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'-\tau(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})'\big)\Big)'-\tau(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})''\Big)N$$

$$+\Big(sin\theta+cos\theta\Big)\Big(\big(2cos\theta-cos^2\theta\big)\tau\big(\kappa(\frac{\tau}{\sqrt{\kappa^2+\tau^2}})'-\tau(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}})'\big)$$

$$+\big(sin2\theta+cos^2\theta\big)\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'''\Big)B$$

bulunur. $D_{T_\gamma}C_\gamma,\ D^2_{T_\gamma}C_\gamma,\ D^3_{T_\gamma}C_\gamma$ türevlerindeki $T,\ N,\ B$ vektörleri de α eğrisinin Frenet aparatları türünden hesaplanırsa,

$$
\begin{aligned}
T &= \frac{\kappa\sqrt{\kappa^2+\tau^2}}{\kappa\tau'-\kappa'\tau}D_BC-\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\frac{\kappa^2+\tau^2}{\kappa\tau'-\kappa'\tau}C\,,\\
N &= \frac{sin\theta\,\sqrt{\kappa^2+\tau^2}}{\big(1-cos\theta\big)\big(\kappa\tau'-\kappa'\tau\big)}D^2_BC\\
&\quad+\frac{sin\theta\,\big(\kappa^2+\tau^2\big)}{\big(cos\theta-1\big)\big(\kappa\tau'-\kappa'\tau\big)^2}\Big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''-\tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''\Big)D_BC\\
&\quad+\frac{sin\theta\,\big(\kappa^2+\tau^2\big)^{3/2}}{\big(cos\theta-1\big)\big(\kappa\tau'-\kappa'\tau\big)^2}\Bigg(\frac{\big(\kappa/\sqrt{\kappa^2+\tau^2}\big)'}{\big(\tau/\sqrt{\kappa^2+\tau^2}\big)'}\Bigg)'\Big(\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'\Big)^2C,\\
B &= \frac{-\tau\sqrt{\kappa^2+\tau^2}}{\kappa\tau'-\kappa'\tau}D_BC-\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'\frac{\kappa^2+\tau^2}{\kappa\tau'-\kappa'\tau}C
\end{aligned}
$$

bulunur. Son olarak (4.2.24) denkleminin $\eta_{\gamma1}$, $\eta_{\gamma2}$ ve $\eta_{\gamma3}$ katsayıları

$$\vartheta=\tau\sqrt{\lambda^2+\mu^2}\,,\qquad \parallel W_\gamma\parallel=\frac{\sqrt{\kappa^2+\tau^2}}{\tau\sqrt{\lambda^2+\mu^2}}\qquad \text{ve}\qquad \theta_\gamma=arctan(\frac{1}{\mu\kappa-\lambda\tau})$$

bağıntıları kullanılarak α eğrisinin Frenet aparatları türünden yazılır ve bunlar sırasıyla, $\eta_1,\ \eta_2,\ \eta_3$ ile gösterilirse

$$\eta_1 = -\frac{\big(arctan(\frac{1}{\mu\kappa-\lambda\tau})\big)''}{\big(arctan(\frac{1}{\mu\kappa-\lambda\tau})\big)'} - \frac{\Big(\sqrt{\kappa^2+\tau^2}\big(arctan(\frac{1}{\mu\kappa-\lambda\tau})\big)'\Big)'}{\sqrt{\kappa^2+\tau^2}\big(arctan(\frac{1}{\mu\kappa-\lambda\tau})\big)'} \ ,$$

$$\eta_2 = \kappa^2+\tau^2+\Big(\big(arctan(\frac{1}{\mu\kappa-\lambda\tau})\big)'\Big)^2 - \Big(\frac{\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)''}{\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'}\Big)'$$

$$+\frac{\Big(\sqrt{\kappa^2+\tau^2}\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\Big)'}{\sqrt{\kappa^2+\tau^2}\Big(\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\Big)^2}\Big(arctan(\frac{1}{\mu\kappa-\lambda\tau})\Big)'',$$

$$\eta_3 = \Big(\big(\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\big)^2\Big)' - \frac{\Big(\sqrt{\kappa^2+\tau^2}\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'\Big)'}{\sqrt{\kappa^2+\tau^2}}\big(arctan\frac{1}{\mu\kappa-\lambda\tau}\big)'$$

şeklinde bulunur. Bu katsayılar (4.2.24) bağıntısında yerine yazılırsa, γ partner eğrisinin diferensiyel denklemi α eğrisinin Frenet aparatları türünden elde edilir ve böylece ispat tamamlanır. $\square$

4.2.5 Bertrand Partner Eğrisinin Darboux Vektörüne Göre Harmonikliği

(3.2.2) ve (3.2.4) bağıntılarında H ortalama eğrilik vektörünün yerine Darboux vektörü W yazılırsa, biharmonik eğrilerin sınıflandırılması Darboux vektörüne göre yeniden yorumlanabilir. Buna göre aşağıdaki teoremleri verelim.

Teorem 4.2.15 (α, γ) Bertrand eğri çifti olsun. γ partner eğrisinin konneksiyona göre harmoniklik koşulları, W_γ Darboux vektörü türünden aşağıdaki önermelerle verilir:
1. γ partner eğrisinin Darboux vektörü biharmoniktir ancak ve ancak

$$\tau_\gamma'' = 0, \quad \vartheta\kappa_\gamma\tau_\gamma' - \vartheta\kappa_\gamma'\tau_\gamma = 0\,, \quad \kappa_\gamma'' = 0.$$

2. γ partner eğrisinin Darboux vektörü 1.tipten harmoniktir ancak ve ancak

$$\tau_\gamma'' = -\lambda_0\tau_\gamma, \quad \vartheta\kappa_\gamma\tau_\gamma' - \vartheta\kappa_\gamma'\tau_\gamma = 0\,,\ \kappa_\gamma'' = -\lambda_0\kappa_\gamma\,, \quad \lambda_0 \in \mathbb{R}.$$

İspat. Lemma 4.0.9 ve (4.2.23) bağıntısından ΔW_γ Darboux vektörünün Laplace görüntüsü

$$\begin{aligned} \Delta W_\gamma &= -D^2_{T_\gamma} W_\gamma \\ &= \tau_\gamma'' T_\gamma + \big(\vartheta\kappa_\gamma\tau_\gamma' + \vartheta\kappa_\gamma'\tau_\gamma\big)N_\gamma + \kappa_\gamma'' B_\gamma \end{aligned}$$

dir. $\Delta W_\gamma = 0$ ise birinci önerme, $\Delta W_\gamma = \lambda_0 W_\gamma$ ise ikinci önermesi sağlanır. $\square$

Sonuç 4.2.3 (α, γ) bir Bertrand eğri çifti ve T_γ ile T vektörleri arasındaki açı θ olsun. Konneksiyona göre γ partner eğrisi biharmoniktir ancak ve ancak $tan\theta = -1$ dir.

İspat. (3.2.2) bağıntısında H nın yerine Darboux vektörü

$$W_\gamma = \frac{1}{\tau\sqrt{\lambda^2 + \mu^2}} W$$

yazılarak bu vektörün Laplace görüntüsü hesaplanırsa,

$$\Delta W_\gamma = \frac{sin\theta(cos\theta + sin\theta)}{\tau(\lambda^2+\mu^2)} D_B^2 W + \frac{2sin\theta(cos\theta + sin\theta)}{\lambda^2+\mu^2}(\frac{1}{\tau})' D_B W$$

$$+\frac{(cos\theta + sin\theta)^2}{\lambda^2+\mu^2}(\frac{1}{\tau})'' W + \frac{cos\theta(cos\theta + sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\tau''\tau - 2(\tau')^2\big)T$$

$$+\frac{cos\theta(cos\theta + sin\theta)}{\tau(\lambda^2+\mu^2)}\big(\kappa\tau' - \kappa'\tau\big)N + \frac{cos\theta(cos\theta + sin\theta)}{\tau^2(\lambda^2+\mu^2)}\big(\kappa''\tau - 2\kappa'\tau'\big)B$$

olur. $\Delta W_\gamma = 0$ ise, $cos\theta + sin\theta = 0$ veya $tan\theta = -1$ elde edilir. □

Teorem 4.2.16 (α, γ) bir Bertrand eğri çifti olsun. γ partner eğrisinin konneksiyona göre harmoniklik koşulları, C_γ birim Darboux vektörü türünden aşağıdaki önermelerle verilir:

1. γ partner eğrisinin birim Darboux vektörü biharmoniktir ancak ve ancak

$$\frac{(\theta_\gamma)''}{(\theta_\gamma)'} = 0, \quad \big(\theta'_\gamma\big)^2 = 0\,, \quad \theta'_\gamma\vartheta \parallel W_\gamma \parallel = 0.$$

2. γ partner eğrisinin birim Darboux vektörü 1.tipten harmoniktir ancak ve ancak

$$\frac{\theta''_\gamma}{\theta'_\gamma} = 0, \quad \big(\theta'_\gamma\big)^2 = \lambda_0\,, \quad \theta'_\gamma\vartheta \parallel W_\gamma \parallel = 0, \quad \lambda_0 \in \mathbb{R}.$$

İspat. Lemma 4.0.10 dan C_γ birim Darboux vektörünün Laplace görüntüsü

$$\Delta C_\gamma = -D^2_{T_\gamma} C_\gamma$$

$$= \big(\frac{-\phi''_\gamma}{\phi'_\gamma}\big) D_{T_\gamma} C_\gamma + \big(\phi'_\gamma\big)^2 C_\gamma - \phi'_\gamma\vartheta \parallel W_\gamma \parallel N_\gamma$$

olur. $\Delta C_\gamma = 0$ ise birinci önerme, $\Delta C_\gamma = \lambda_0 C_\gamma$ ise ikinci önerme sağlanır. □

Teorem 4.2.16 da verilen γ partner eğrisinin biharmonik olma koşulu, α eğrisinin Frenet aparatları türünden yazılırsa, aşağıdaki sonuç verilebilir.

Sonuç 4.2.4 (α,γ) bir Bertrand eğri çifti ve T_γ ile T vektörleri arasındaki açı θ olsun. Konneksiyona göre γ partner eğrisi biharmoniktir ancak ve ancak $tan\theta=-1$ dir.

İspat. (3.2.2) bağıntısında H ortalama eğrilik vektörü yerine birim Darboux vektörü $C_\gamma=C$ yazılır ve bu vektörün Laplace dönüşümü hesaplanırsa,

$$\begin{aligned}
\Delta C_\gamma &= -D^2_{T_\gamma}C_\gamma \\
&= -\Big(sin\theta\big(sin\theta+cos\theta\big)\Big)D^2_B C-\Big(cos\theta\big(sin\theta+cos\theta\big)\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)''\Big)T \\
&\quad -\Big(cos\theta\big(sin\theta+cos\theta\big)\big(\kappa\big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\big)'-\tau\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)'\big)\Big)N \\
&\quad -\Big(cos\theta\big(sin\theta+cos\theta\big)\big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\big)''\Big)B
\end{aligned}$$

olur. $\Delta C_\gamma=0$ ise, $cos\theta+sin\theta=0$ veya $tan\theta=-1$ elde edilir. $\square$

Sonuç 4.2.5 (α,γ) bir Bertrand eğri çifti olsun. α eğrisi dairesel helis ise, γ partner eğrisi W_γ Darboux vektörüne göre biharmonik bir eğridir.

İspat. α eğrisi dairesel helis alınırsa, γ partner eğrisi de dairesel helis olur. Buna göre Darboux vektörünün Laplace görüntüsü $\Delta W_\gamma=-D^2_{T_\gamma}W_\gamma$ eşitliğinden

$$\Delta W_\gamma=-\tau''_\gamma T_\gamma+\big(\vartheta\kappa'_\gamma\tau_\gamma-\vartheta\kappa_\gamma\tau'_\gamma\big)N_\gamma-\kappa''_\gamma B_\gamma$$

olur ve Teorem 4.2.15 ten $\tau''_\gamma=0$, $\vartheta\kappa_\gamma\tau'_\gamma-\vartheta\kappa'_\gamma\tau_\gamma=0$ ve $\kappa''_\gamma=0$ elde edilir. $\square$

Örnek 4.2.2 (α,γ) Bertrand eğri çifti olsun. Öyle ki α eğrisinin eğrilikleri κ ve τ

$$\lambda=\frac{(\frac{\tau}{\kappa})'}{(\frac{\kappa^2+\tau^2}{\tau})'} \quad \text{ve} \quad \mu=\frac{(\frac{\kappa^2+\tau^2}{\tau})'-(\frac{\kappa}{\tau})'\kappa}{(\frac{\kappa^2+\tau^2}{\tau})'\tau}$$

koşulunu sağlasın [30]. Bu şartlar altında $\lambda\kappa+\mu\tau=1$ olup, γ partner eğrisi bir genel helistir ve bu eğriyi karakterize eden diferensiyel denklemler

Teorem 4.2.11 e göre:

$$\vartheta\Big(\big(\kappa_\gamma\big)''\tau_\gamma-\kappa_\gamma\big(\tau_\gamma\big)''\Big)^2 D_{T_\gamma}W_\gamma+\vartheta\Big(\big(\kappa_\gamma\big)''\tau_\gamma-\kappa_\gamma\big(\tau_\gamma\big)''\Big)W_\gamma=0$$

Teorem 4.2.12 ye göre:

$$\Big((\frac{\mu\kappa-\lambda\tau}{\tau(\lambda^2+\mu^2)})''\frac{1}{\tau(\lambda^2+\mu^2)}-\frac{\mu\kappa-\lambda\tau}{\tau(\lambda^2+\mu^2)}(\frac{1}{\tau(\lambda^2+\mu^2)})''\Big)^2\Big(\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\Big)D_BW$$

$$+\Bigg(\Big((\frac{\mu\kappa-\lambda\tau}{\tau(\lambda^2+\mu^2)})''\frac{1}{\tau(\lambda^2+\mu^2)}-\frac{\mu\kappa-\lambda\tau}{\tau(\lambda^2+\mu^2)}(\frac{1}{\tau(\lambda^2+\mu^2)})''\Big)^2\Big(\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\Big)(\frac{-\tau'}{\tau})$$

$$+\Big((\frac{\mu\kappa-\lambda\tau}{\tau(\lambda^2+\mu^2)})''\frac{1}{\tau(\lambda^2+\mu^2)^{3/2}}-\frac{\mu\kappa-\lambda\tau}{\tau(\lambda^2+\mu^2)^{3/2}}(\frac{1}{\tau(\lambda^2+\mu^2)})''\Big)\Bigg)W=0$$

şeklinde verilir.

5. TARTIŞMA ve SONUÇ

Bu tez çalışmasında önce üç boyutlu Öklid uzayında keyfi parametreyle verilen bir eğrinin diferensiyel denklemleri eğrinin teğet vektörüne ve Darboux vektörüne göre yazıldı. Eğrinin harmoniklik koşulları ise sırasıyla Laplace operatörüne göre (biharmonik veya 1.tipten harmonik) ve normal Laplace operatörüne göre (zayıf biharmonik veya 1.tipten harmonik) verildi. Sonra eğrinin bir involütü tanımlanarak, involüt eğrisinin diferensiyel denklemleri ve harmoniklik şartları involüt eğrisinin kendi parametresine göre yazıldı. Daha sonra evolüt involüt eğrilerinin Frenet çatıları arasındaki geçiş matrisleri kullanılarak, involüt eğrisi için yazılan diferensiyel denklemler ve harmoniklik koşulları esas eğrinin Frenet aparatları türünden ifade edildi. Benzer şekilde esas eğrinin bir Bertrand eğrisi olması halinde, bu eğrinin bir Bertrand partneri tanımlandı ve partner eğrisinin karakterizasyonları involüt eğrisinde olduğu gibi önce kendi parametresine göre yazıldı. Yine bağlantılı eğrilerin Frenet çatıları arasındaki geçişler kullanılarak Bertrand partner eğrisine ait denklemler ve harmoniklik şartları esas eğrinin Frenet aparatları türünden ifade edildi. Son olarak teoremlere ait sonuçlar verildi ve sayısal bir örnekle bu bölümdeki iddialar desteklendi.

Mannheim eğrilerinin karakterizasyonları bu çalışmada kullanılan yöntem ve teknikle hesaplanarak özgün bir yayın yapılması düşünülmektedir. Tezde kullanılan yöntem, Öklid dışı uzaylarda veya Frenet çatısından farklı yapılarda çalışmalar yapmak isteyen bilim insanlarına yardımcı olacaktır.

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