T. C.

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FEN BİLİMLERİ ENSTİTÜSÜ

İNVOLÜT EĞRİSİNİN VEKTÖREL MOMENT EĞRİLERİ ÜZERİNE

SEZER DURMAZ

YÜKSEK LİSANS TEZİ

MATEMATİK ANABİLİM DALI

ORDU 2021

TEZ BİLDİRİMİ

Tez yazım kurallarına uygun olarak hazırlanan ve kullanılan intihal tespit programının sonuçlarına göre; bu tezin yazılmasında bilimsel ahlak kurallarına uyulduğunu, başkalarının eserlerinden yararlanılması durumunda bilimsel normlara uygun olarak atıfta bulunulduğunu, tezin içerdiği yenilik ve sonuçların başka bir yerden alınmadığını, kullanılan verilerde herhangi bir tahrifat yapılmadığını, tezin herhangi bir kısmının bu üniversite veya başka bir üniversitedeki başka bir tez çalışması olarak sunulmadığını beyan ederim.

SEZER DURMAZ

Not: Bu tezde kullanılan özgün ve başka kaynaktan yapılan bildirişlerin, çizelge, şekil ve fotoğrafların kaynak gösterilmeden kullanımı, 5846 sayılı Fikir ve Sanat Eserleri Kanunundaki hükümlere tabidir.

ÖZET

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MATEMATİK ANABİLİM DALI

YÜKSEK LİSANS TEZİ, 77 SAYFA

TEZ DANIŞMANI: DR. ÖĞRETİM ÜYESİ SÜLEYMAN ŞENYURT

Bu çalışmada; ilk olarak birim hızlı herhangi bir eğrinin involütünün Frenet vektörleri ve Darboux vektörüne ait vektörel moment vektörleri tanımlandı. Daha sonra moment vektörlerinin çizdiği eğrilerin Frenet vektörleri ve eğrilikleri ayrı ayrı hesaplanarak esas eğri ile aralarında bağlantılar verildi. Son olarak vektörel moment eğrileriyle, involüt eğrisinin sabit genişlikli eğri kategorisinde olup olmadıkları araştırıldı.

Anahtar Kelimeler: İnvolüt Eğrisi, Vektörel Moment Vektörü, Sabit Genişlikli Eğri, Darboux.

ABSTRACT

ON THE VECTORIAL MOMENTS OF INVOLUTE CURVE

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In this research, firstly, the vectorial moments are defined based on the Frenet and Darboux vectors of the involute of any given unit speed curve. Then, the Frenet vectors and curvatures of the curves sketched by vectorial moments are calculated and the corresponding relations are given. Finally, it was examined if each produced vectorial moment curves and the involute curve are in the class of constant width curves.

Keywords: Involute Curve, Vectorial Moment, Constant-Width Curve Pairs, Darboux Vector.

TEŞEKKÜR

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Aynı zamanda, maddi ve manevi desteklerini her an üzerimde hissettiğim annem Ayşe Durmaz'a ve kardeşim Emrah Durmaz'a teşekkürü borç bilirim.

İÇİNDEKİLER

ŞEKİL LİSTESİ

SİMGELER ve KISALTMALAR LİSTESİ

$\boldsymbol{E^3}$	:	3-Boyutlu Öklid Uzayı
$\Vert . \Vert$	:	Norm
$\wedge$	:	Vektörel Çarpım
$\boldsymbol{\alpha}^*$	:	İnvolüt Eğrisi
$\boldsymbol{T}^*$	:	İnvolüt Eğrisinin Teğeti
$\boldsymbol{N}^*$	:	İnvolüt Eğrisinin Asli Normali
$\boldsymbol{B}^*$	:	İnvolüt Eğrisinin Binormali
$\boldsymbol{\kappa_1}$	:	İnvolüt Eğrisine Ait Eğrilik
$\boldsymbol{\tau_1}$	:	İnvolüt Eğrisine Ait Torsiyon
$\boldsymbol{C}^*$	:	İnvolüt Eğrisine Ait Birim Darboux Vektörü
$\boldsymbol{V_1}$	:	T^* Vektörüne Ait Vektörel Moment Vektörü
$\boldsymbol{V_2}$	:	N^* Vektörüne Ait Vektörel Moment Vektörü
$\boldsymbol{V_3}$	:	B^* Vektörüne Ait Vektörel Moment Vektörü
$\boldsymbol{V_4}$	:	C^* Vektörüne Ait Vektörel Moment Vektörü
$\boldsymbol{\beta_1}$	:	V_1 Vektörel Moment Vektörünün Çizdiği Eğri
$\boldsymbol{\beta_2}$	:	V_2 Vektörel Moment Vektörünün Çizdiği Eğri
$\boldsymbol{\beta_3}$	:	V_3 Vektörel Moment Vektörünün Çizdiği Eğri
$\boldsymbol{\beta_4}$	:	V_4 Vektörel Moment Vektörünün Çizdiği Eğri
$\boldsymbol{T_{\beta_1}}$	:	β_1 Eğrisinin Teğeti
$\boldsymbol{N_{\beta_1}}$	:	β_1 Eğrisinin Asli Normali
$\boldsymbol{B_{\beta_1}}$	:	β_1 Eğrisinin Binormali
$\boldsymbol{\kappa_{\beta_1}}$	:	β_1 Eğrisinin Eğriliği
$\boldsymbol{\tau_{\beta_1}}$	:	β_1 Eğrisinin Torsiyonu
$\boldsymbol{T_{\beta_2}}$	:	β_2 Eğrisinin Teğeti
$\boldsymbol{N_{\beta_2}}$	:	β_2 Eğrisinin Asli Normali
$\boldsymbol{B_{\beta_2}}$	:	β_2 Eğrisinin Binormali
$\boldsymbol{\kappa_{\beta_2}}$	:	β_2 Eğrisinin Eğriliği
$\boldsymbol{\tau_{\beta_2}}$	:	β_2 Eğrisinin Torsiyonu
$\boldsymbol{T_{\beta_3}}$	:	β_3 Eğrisinin Teğeti
$\boldsymbol{N_{\beta_3}}$	:	β_3 Eğrisinin Asli Normali
$\boldsymbol{B_{\beta_3}}$	:	β_3 Eğrisinin Binormali
$\boldsymbol{\kappa_{\beta_3}}$	:	β_3 Eğrisinin Eğriliği
$\boldsymbol{\tau_{\beta_3}}$	:	β_3 Eğrisinin Torsiyonu
$\boldsymbol{T_{\beta_4}}$	:	β_4 Eğrisinin Teğeti
$\boldsymbol{N_{\beta_4}}$	:	β_4 Eğrisinin Asli Normali
$\boldsymbol{B_{\beta_4}}$	:	β_4 Eğrisinin Binormali
$\boldsymbol{\kappa_{\beta_4}}$	:	β_4 Eğrisinin Eğriliği

$\boldsymbol{\tau_{\beta_4}}$	:	β_4 Eğrisinin Torsiyonu
$\boldsymbol{det}$	:	Determinant

1. GİRİŞ

İnvolüt eğrileri üzerinde çok sayıda çalışma mevcuttur. Ancak ilk defa involüt eğri düşüncesi optik çalışmalar yapılırken keşfedilmiştir (Huygens, 1658).

3-Boyutlu Öklid uzayında regüler herhangi bir eğrinin karakterizasyonları ile ilgili birçok çalışma yapılmıştır (Chen, 2001). "Constant ratio Hypersurface" adlı çalışmasında eğriyi Frenet vektölerinin lineer birleşimi olarak yazmıştır. Daha sonra katsayılar arasındaki bağlantıları vermiştir.

Herhangi iki eğrinin karşılıklı noktalarındaki teğetleri arasında farklı teoriler geliştirilmiştir. Bunlardan biri de (Akdoğan ve Mağden, 2001). "Some Characterization of Curves of Constant Breadth in E^n Space" isimli çalışmasıdır. Bu çalışmada sabit genişlikli eğri kavramını vermişlerdir.

Yaz (2005), "Sabit Genişlikli Eğrilerin Kinematiği" isimli doktora tezinde düzlemde ve uzayda sabit genişlikli eğrileri ifade ederek bazı karakterizasyonlar elde etmişlerdir.

Şardağ (2019), "Alternatif Çatının Vektörel Moment Eğrileri Üzerine" isimli çalışmasında alternatif çatı vektörlerinin vektörel moment vektörlerinin çizdiği eğrilerin Frenet Vektörleri, eğrilik ve torsiyonunu hesaplamıştır. Elde edilen vektörel moment eğrilerinin sabit genişlikli eğri kategorisine dahil olup olmadığını araştırmıştır.

Tunçer (2017), "Vectorial moments of curves in Euclidean 3-space" isimli çalışmasında Frenet vektörlerinin vektörel moment vektörlerini ve bu vektörlerin çizdiği eğrinin Frenet vektörlerini, eğriliklerini ve torsiyonunu göstermiştir.

Akdoğan (1994), "E^n Sabit Genişlikli Eğriler" isimli doktora tezinde E^n Öklid uzayında sabit genişlikli eğriler için oluşturulan diferansiyel denklem sistemi, yaklaşım metodu ile çözülerek bazı karakterizasyonlar vermiştir.

Bu çalışmada herhangi bir eğrinin involütünün Frenet vektörleri ve Darboux vektörünün vektörel moment vektörleri bulundu. Daha sonra bu vektörlerin çizdiği eğrilerin Frenet aparatları hesaplanarak esas eğrinin Frenet vektörleri cinsinden ifadeleri verildi.

2. GENEL BİLGİLER

2.1 Öklid Uzayı

Bu bölümde, 3-boyutlu Öklid uzayı ile ilgili temel kavramlara yer verilmiştir. A boştan farklı bir cümle ve V de K cismi üzerinde bir vektör uzayı olsun.

$$f : A \times A \to V$$

fonksiyonu aşağıdaki aksiyomları sağlarsa A'ya V ile birleştirilmiş bir afin uzay denir:

$$A_1 : \forall P, Q, R \in A \text{ için } f(P,Q) + f(Q,R) = f(P,R)$$

$$A_2 : \forall P \in A, \forall \alpha \in V \text{için } f(P,Q) = \alpha$$

olacak şekilde bir tek $Q \in A$ noktası vardır. A, V ile birleşen bir afin uzay olsun. $P_0, P_1, P_2, P_3 \in A$ noktaları için $\{P_0P_1, P_0P_2, P_0P_3\}$ cümlesi V nin bir bazı ise $\{P_0, P_1, P_2, P_3\}$ nokta 4-lüsüne bir afin çatısı denir. Burada P_0 noktasına çatının başlangıç noktası ve P_i, $1 \leq i \leq 3$, noktalarına da çatının birim noktaları denir. $boyV = 3$ ise A ya 3-boyutlu bir afin uzay denir.

$$\langle , \rangle : V \times V \to \mathbb{R}$$

şeklinde tanımlı fonksiyon aşağıdaki aksiyomları sağlarsa bu fonksiyona bir iç çarpım fonksiyonu denir: $\forall x, y, z \in V$, $\forall a, b \in \mathbb{R}$ için

a. Bilineerlik Aksiyomu;

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle, \langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle,$$

b. Simetri Aksiyomu;

$$\langle x, y \rangle = \langle y, x \rangle,$$

c. Pozitif Tanımlılık (kararlılık) Aksiyomu;

$$\langle x, x \rangle \geq 0, \qquad \langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

$\mathbb{R}^3$ afin uzay, $\forall\ X, Y \in \mathbb{R}^3$ olsun,

$$\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3$$

şeklinde tanımlı fonksiyon bir iç çarpım fonksiyonudur. Bu fonksiyona standart iç çarpım veya Öklid iç çarpımı denir. Üzerinde Öklid iç çarpımı tanımlı $\mathbb{R}^3$ afin uzayına Öklid uzayı denir ve $\mathbb{E}^3$ ile gösterilir. $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3$ olmak üzere,

$$d \ : \ \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$

$$(X, Y) \to d(X, Y) = \sqrt{\sum_{i=1}^{3}(x_i - y_i)^2}$$

şeklinde tanımlanan d fonksiyonuna uzaklık fonksiyonu ve $d(X, Y)$ reel sayısına da X ve Y noktaları arasındaki uzaklık denir. $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ diferensiyellenebilir fonksiyona $\mathbb{R}^3$ te bir eğri denir. Burada I aralığına α eğrisinin parametre aralığı, $s \in I$ değişkenine de α eğrisinin parametresi denir.

α birim hızlı eğrisinin teğet, asli normal ve binormal vektörleri sırasıyla

$$T(s) = \alpha'(s), \quad N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad B(s) = T(s) \wedge N(s)$$

şeklinde bulunur. Bu vektörlere Frenet vektörleri denir. α birim hızlı eğri değil ise Frenet vektörleri

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \quad N(s) = B(s) \wedge T(s), \qquad B(s) = \frac{\alpha'(s) \wedge \alpha''(s)}{\|\alpha'(s) \wedge \alpha''(s)\|} \qquad (2.1.1)$$

şeklinde verilir (Hacısalihoğlu, 1983).
α eğrisi birim hızlı ve Frenet vektörleri T, N, B olsun.

$$\kappa : I \to \mathbb{R}, \quad \kappa(s) = \|T'(s)\|$$

şeklinde tanımlı fonksiyona α eğrisinin eğrilik fonksiyonu, $\kappa(s)$ sayısına eğrinin $\alpha(s)$

noktasındaki eğriliği denir.

$$\tau : I \to \mathbb{R}, \quad \tau(s) = -\langle B'(s), N(s) \rangle$$

şeklinde tanımlı fonksiyona α eğrisinin burulma fonksiyonu, $\tau(s)$ sayısına da eğrinin $\alpha(s)$ noktasındaki burulması denir (Sabuncuoğlu, 2014). Eğer eğri keyfi parametre ile verilmiş ise κ eğriliği ve τ burulması sırasıyla

$$\kappa(s) = \frac{\|\alpha'(s) \wedge \alpha''(s)\|}{\|\alpha'(s)\|^3}, \quad \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\|\alpha'(s) \wedge \alpha''(s)\|^2} \tag{2.1.2}$$

denklemleriyle verilir (Hacısalihoğlu, 1983). $\alpha : I \to \mathbb{R}^3$ eğrisinin Frenet formülleri

$$T'(s) = \kappa(s)N(s), \quad N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \quad B'(s) = -\tau(s)N(s) \tag{2.1.3}$$

bağıntısıyla verilir (Hacısalihoğlu, 1983). α eğrisi Frenet vektörlerinin lineer birleşimi olarak

$$\alpha(s) = f(s)T(s) + g(s)N(s) + h(s)B(s) \tag{2.1.4}$$

şeklinde yazılır. Burada f,g ve h fonksiyonları arasında

$$f^{'}(s) = 1 + g(s)\kappa(s), \quad g^{'}(s) = h(s)\tau(s) - f(s)\kappa(s), \quad h^{'}(s) = -g(s)\tau(s) \tag{2.1.5}$$

bağıntısı vardır (Chen, 2001). α eğrisi üzerinde herhangi bir $\vec{x}$ vektörünün $\vec{x^*}$ vektörel moment vektörü

$$\vec{x^*} = \vec{\alpha} \wedge \vec{x} \tag{2.1.6}$$

şeklinde tanımlanır. Frenet vektörlerinin vektörel moment vektörleri sırasıyla

$$T^* = \alpha(s) \wedge T(s), \quad N^* = \alpha(s) \wedge N(s), \quad B^* = \alpha(s) \wedge B(s)$$

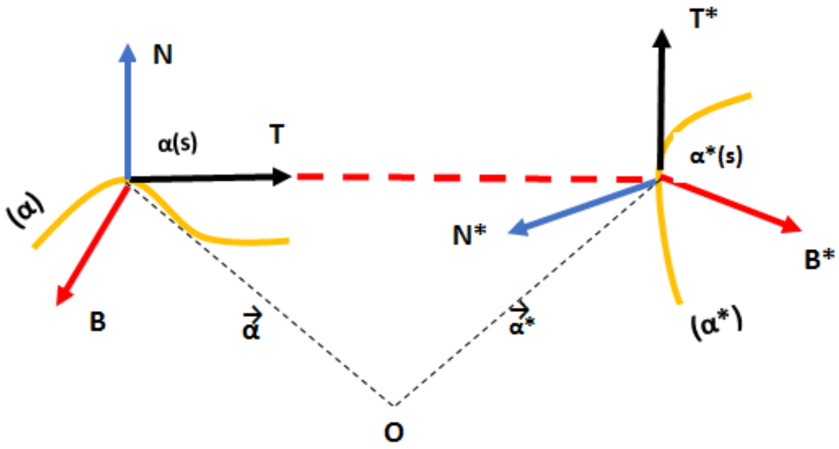
şeklinde bulunur.

2.2 Öklid Uzayında İnvolüt Eğrisi

Tanım 2.2.1 Birim hızlı $\alpha : I \to R^3$ eğrisi ile aynı aralıkta tanımlı $\alpha^*(s)$ eğrilerinin Frenet vektörleri sırasıyla $T(s), N(s), B(s)$ ve $T^*(s), N^*(s), B^*(s)$ olsun. Her bir $s \in I$ için $\alpha(s)$ noktasındaki teğeti $\alpha^*(s)$ noktasından geçiyor ve

$$< T(s), T^*(s) >= 0$$

ise $\alpha^*(s)$ eğrisine $\alpha(s)$ eğrisinin bir involütü denir (Sabuncuoğlu, 2006), (Şekil 2.1).



Şekil 2.1: İnvolüt Eğrisi

Teorem 2.2.1 $\alpha^*(s)$ eğrisi $\alpha(s)$ eğrisinin bir involütü olsun. Bu eğriler arasında

$$\alpha^*(s) = \alpha(s) + \lambda(s)T(s), \quad \lambda(s) = c - s, \quad c \in \Re \tag{2.2.1}$$

bağıntısı vardır (Sabuncuoğlu, 2006).

İspat. $\alpha^*(s)$ eğrisinin s parametresine göre türevi alındığında

$$(\alpha^*(s))' = \alpha'(s) + \lambda'(s)T(s) + \lambda(s)\kappa(s)N(s)$$

olur. Burada gerekli düzenlemeler yapıldığında

$$(\alpha^*(s))' = (1 + \lambda'(s))T + \lambda(s)\kappa(s)N(s)$$

şeklinde olur. α^* eğrisi α eğrisinin involütü olduğundan

$$< (\alpha^*(s))', T(s) >= 0$$

$(\alpha^*(s))'$ bağıntısı burada yerine yazıldığında

$$< (1 + \lambda'(s))T(s) + \lambda(s)\kappa(s)N(s), T(s) >= 0$$

olur. İç çarpım işlemi yapıldığında

$$1 + \lambda'(s) = 0$$

halini alır. Elde edilen bağıntının s parametresine göre integrali alındığında

$$\lambda(s) = c - s, \quad c \in \Re$$

şeklinde olur. $\square$

Teorem 2.2.2 $\alpha^*(s)$ eğrisi, $\alpha(s)$ eğrisinin bir involütü ve Frenet vektörleri sırasıyla $\{T^*(s), N^*(s), B^*(s)\}$, $\{T(s), N(s), B(s)\}$ olsun. Bu vektörler

$$\begin{aligned}
T^*(s) &= N(s), \\
N^*(s) &= -\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}T(s) + \frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}B(s), \\
B^*(s) &= \frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}T(s) + \frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}B(s)
\end{aligned}$$

bağıntısıyla verilir. Burada $\{\kappa(s), \tau(s)\}$, $\alpha(s)$ eğrisinin eğriliği ve torsiyonudur (Sabuncuoğlu, 2006).

İspat. $\alpha^*(s)$ eğrisinin türevi alındığında

$$(\alpha^*(s))' = (c-s)\kappa(s)N(s)$$

olur. Buradan norm alındığında

$$\| (\alpha^*(s))' \| = | (c-s)\kappa(s)$$

bulunur. (2.1.1) bağıntısından

$$T^*(s) = \frac{(c-s)\kappa(s)}{| (c-s)\kappa(s) |}N$$

olur. Buradan

$$T^*(s) = N(s) \quad veya \quad T^*(s) = -N(s)$$

bulunur. $T^*(s) = N(s)$ için

$$\| \alpha^*(s)' \| = |(c-s)\kappa(s)|$$

$\alpha^*(s)$ eğrisinin ikinci türevi alındığında

$$\begin{aligned}
\alpha^*(s)'' &= -\kappa(s)N(s) + (c-s)\kappa'(s)N(s) + (c-s)\kappa(s)(-\kappa(s)T(s) + \tau(s)B(s)) \\
\alpha^*(s)'' &= -(c-s)\kappa^2(s)T(s) + \big(-\kappa(s) + (c-s)\kappa'(s)\big)N(s) + (c-s)\kappa(s)\tau(s)B(s)
\end{aligned}$$

olur. Bu ifadenin bir kez daha türevi alınırsa

$$\begin{aligned}
\alpha^*(s)''' &= -\kappa^2 T - (c-s)2\kappa\kappa' T - (c-s)\kappa^2\kappa N \\
&\quad +\big(-\kappa' - \kappa' + (c-s)\kappa''\big)N + \big(-\kappa + (c-s)\kappa'\big)(-\kappa T + \tau B) \\
&\quad -\kappa\tau B + (c-s)\kappa'\tau B + (c-s)\kappa\tau' B + (c-s)\kappa\tau(-\tau N) \\
&= \big(-\kappa^2 - 3(c-s)\kappa\kappa'\big)T + \big(-2\kappa' + (c-s)(\kappa'' - \kappa^3 - \kappa\tau^2)\big)N \\
&\quad +\big(-2\kappa\tau + (c-s)(2\kappa'\tau + \kappa\tau')\big)B
\end{aligned}$$

ifadesi bulunur. $\alpha^*(s)$ eğrisinin birinci ve ikinci türevlerinden

$$(\alpha^*(s))' \wedge (\alpha^*(s))'' = (c-s)^2\kappa^2(s)\tau(s)T(s) + (c-s)^2\kappa^3(s)B(s)$$

olur. Bu ifadenin normu alındığında

$$\| (\alpha^*(s))' \wedge (\alpha^*(s))'' \| = \sqrt{(c-s)^4\kappa^4(s)\tau^2(s) + (c-s)^4\kappa^6(s)}$$

eşitliği bulunur. Burada gerekli düzenlemeler sonunda

$$\| (\alpha^*(s))' \wedge (\alpha^*(s))'' \| = (c-s)^2\kappa^2(s)\sqrt{\kappa^2(s)+\tau^2(s)} \tag{2.2.2}$$

olur. (2.1.1) bağıntısından $\alpha^*(s)$ eğrisinin binormal vektörü

$$B^*(s) = \frac{\tau(s)T(s)+\kappa(s)B(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}$$

şeklinde olur. (2.1.1) bağıntısından $\alpha^*(s)$ eğrisinin aslinormal vektörü

$$N^*(s) = \frac{-\kappa(s)T(s)+\tau(s)B(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}$$

elde edilir. □

Teorem 2.2.3 $\alpha^*(s)$, $\alpha(s)$ eğrisinin bir involütü olsun. $\alpha^*(s)$ eğrisinin eğriliği ve burulması sırasıyla $\{\kappa^*(s), \tau^*(s)\}$ şeklinde gösterilsin. Bu eğrilikler

$$\kappa^*(s) = \frac{\sqrt{\kappa^2(s)+\tau^2(s)}}{\lambda(s)\kappa(s)}, \quad \tau^*(s) = \frac{\kappa(s)}{\lambda(s)\big(\kappa^2(s)+\tau^2(s)\big)}\Big(\frac{\tau(s)}{\kappa(s)}\Big)'$$

bağıntısıyla verilir (Sabuncuoğlu, 2006).

İspat. (2.1.2) bağıntısında (2.2.2) ve eğrinin birinci türevi yerine yazıldığında $\alpha^*(s)$ eğrisinin eğriliği

$$\kappa^*(s) = \frac{\sqrt{\kappa^2(s) + \tau^2(s)}}{(c-s)\kappa(s)}$$

şeklinde olur. Eğrinin torsiyonunu hesaplamak için eğrinin birinci, ikinci ve üçüncü türevlerinden

$$det((\alpha^*(s))', (\alpha^*(s))'', (\alpha^*(s))''') = (c-s)^3\kappa^3(s)(\kappa(s)\tau'(s) - \kappa'(s)\tau(s)))$$

determinantı bulunur. Bu determaninat ve (2.2.2) eşitliği (2.1.2) bağıntsında yerine yazıldığında $\alpha^*(s)$ eğrisinin torsiyonu

$$\tau^*(s) = \frac{\kappa(s)\tau'(s) - \kappa'(s)\tau(s)}{(c-s)\kappa(s)(\kappa^2(s) + \tau^2(s))}$$

elde edilir. □

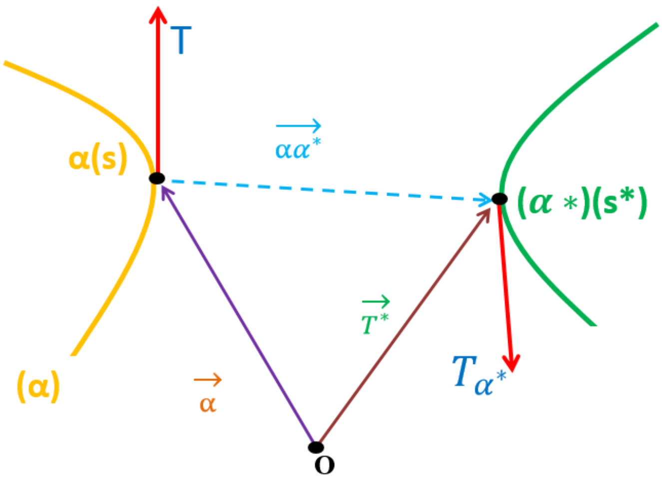
2.3 Sabit Genişlikli Eğri Çifti

Tanım 2.3.1 E^3- Öklid uzayında $\alpha(s)$ ve $\alpha^*(s)$ gibi birim hızlı diferansiyellenebilir iki eğri olsun. Bu eğrilerin $\alpha(s)$ ve $\alpha^*(s^*)$ gibi karşılıklı noktalarında teğetleri paralel ve zıt yönlü, aralarındaki uzaklık sabit ise bu iki eğriye sabit genişlikli eğri çifti denir. Bu eğri çifti $\{\alpha(s), \alpha^*(s)\}$ şeklinde gösterilir (Akdoğan ve Mağden, 2001).

Teorem 2.3.1 Düzlemsel bir $\alpha(s)$ eğrisinin T vektörününün vektörel momenti $T^*(s)$ olsun. $T^*(s)$ vektörünün çizdiği eğri $\alpha^*(s^*)$ ile gösterilirse, bu eğri $\alpha(s)$ eğrisi ile sabit genişlikli eğri çifti oluşturur (Tunçer, 2017).

İspat. $\alpha(s)$ eğrisi Frenet vektörlerinin lineer birleşimi olarak

$$\alpha(s) = f(s)T(s) + g(s)N(s) + h(s)B(s) \tag{2.3.1}$$



Şekil 2.2: Sabit Genişlikli Eğri Çifti

şeklinde yazılır. $T(s)$ vektörünün $T^*(s)$ vektörel moment vektörü

$$\begin{aligned} T^*(s) &= \alpha(s) \wedge T(s) \\ &= (f(s)T(s) + g(s)N(s) + h(s)B(s)) \wedge T(s) \\ &= h(s)N(s) - g(s)B(s) \end{aligned}$$

olur. Bu vektörün çizdiği eğri $\alpha^*(s)$ ile gösterildiğinde

$$\alpha^*(s) = h(s)N(s) - g(s)B(s) \tag{2.3.2}$$

bulunur. $\overrightarrow{\alpha\alpha^*}(s)$ vektörü (Şekil 2.2)'den

$$\overrightarrow{\alpha(s)\alpha^*(s)} = m_1T(s) + m_2N(s) + m_3B(s)$$

yazılır. Buradan

$$\vec{\alpha^*(s)} - \vec{\alpha(s)} = m_1T(s) + m_2N(s) + m_3B(s)$$

olur. Burada (2.3.1) ve (2.3.2) yerlerine yazıldığında

$$h(s)N(s) - g(s)B(s) - f(s)T(s) - g(s)N(s) - h(s)B(s) = m_1T(s) + m_2N(s) + m_3B(s)$$

bulunur. Gerekli düzenlemeler yapıldığında

$$-f(s)T(s)+(h(s)-g(s))N(s)-(g(s)+h(s))B(s)=m_1T(s)+m_2N(s)+m_3B(s)$$

olur. Bu eşitlikten m_1, m_2 ve m_3 katsayıları sırasıyla

$$m_1=-f(s),\qquad m_2=h(s)-g(s),\quad m_3=-(h(s)+g(s))$$

şeklindedir. $\alpha(s)$ eğrisi düzlemsel olduğundan $\tau(s)=0$ olur. $\alpha(s)$ ile $\alpha^*(s)$ eğrisinin aynı düzlemde olması için $f(s)=0$ olmalıdır. Bu katsayılar α^* eğrisinde yerine yazıldığında

$$\alpha^*(s)=\alpha(s)-f(s)T(s)+(h(s)-g(s))N(s)-(h(s)+g(s))B(s) \tag{2.3.3}$$

olur. (2.1.5) bağıntısında $\tau(s)=0$ ve $f(s)=0$ değerleri yerine yazıldığında

$$g'(s)=0\Rightarrow g(s)=sabit\quad ve\quad h'(s)=0\Rightarrow h=sabit$$

eşitliği bulunur. (Şekil 2.2)'de iki nokta arasındaki arasındaki uzaklıktan

$$d(\alpha(s),\alpha^*(s))=\|\overrightarrow{\alpha(s)\alpha^*(s)}\|^2=2h^2(s)+2g^2(s)$$

bağıntısı yazılır. Burada uzaklığın sabit olduğu görülür. (2.3.3) bağıntısında $\alpha^*(s)$ eğrisinin s parametresine göre türevi alındığında

$$\begin{aligned}
\frac{d\alpha^*}{ds^*}.\frac{ds^*}{ds} &= \alpha'+(h-g)'N+(h-g)N'-(h+g)'B-(h+g)B' \\
T_{\alpha^*}.\frac{ds^*}{ds} &= T+(h-g)'N+(h-g)(-\kappa T+\tau B)-(h+g)'B-(h+g)(-\tau N) \\
&= (1-\kappa(h-g))T+((h-g)'+(h+g)\tau)N+((h-g)\tau-(h+g)')B
\end{aligned}$$

bulunur. Burada $g(s) = sbt$, $h(s) = sbt$ ve $\tau(s) = 0$ olma durumu yazıldığında

$$T^* \frac{ds^*}{ds} = (1 - \kappa(h - g))T$$

$$T^* \frac{ds^*}{ds} = \underbrace{(1 + g\kappa}_{f'} - h\kappa)T$$

eşitliği bulunur. $f(s) = 0$ olduğundan

$$T^* \frac{ds^*}{ds} = -h(s)\kappa(s)T(s)$$

olur. $\{\alpha(s), \alpha^*(s)\}$ sabit genişlikli eğri olduğundan $T^*(s) = -T(s)$ olur. Buradan

$$h(s)\kappa(s) = 1 \quad veya \quad h(s) = \frac{1}{\kappa(s)} > 0$$

bulunur. Bu durumda teğet vektörler paralel ve zıt yönlü olur. $\alpha(s)$ ve $\alpha^*(s)$ eğrilerinin aralarındaki uzaklık sabit ve teğet vektörleri paralel-zıt yönlü olduğundan bu iki eğri sabit genişlikli eğri çifti oluşturur. $\square$

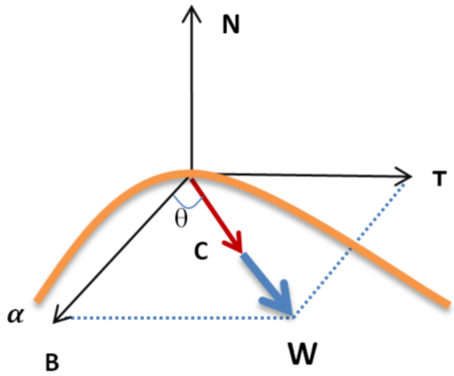
Sonuç 2.3.1 $\alpha(s)$ eğrisinin $N(s)$, $B(s)$ vektörlerinin $N^*(s)$ ve $B^*(s)$ vektörel moment vektörlerinin çizdiği eğriler $\alpha(s)$ eğrisi ile sabit genişlikli eğri çifti oluşturmazlar (Tunçer, 2017).

2.4 Darboux Vektörü

Tanım 2.4.1 Bir eğrinin Frenet çatısı eğri boyunca hareket ederken bir eksen etrafında ani dönme hareketi yapar. Bu eksene Darboux ekseni, Darboux ekseni üzerindeki vektöre de Darboux vektörü adı verilir (Fenchel, 1951), (Şekil 2.3).

Darboux vektörü $W(s)$ olsun. Darboux vektörü

$$W = xT + yN + zB \tag{2.4.1}$$



Şekil 2.3: Birim Darboux Vektörü

şeklinde yazılır.

$$W \wedge T = T', \qquad W \wedge N = N', \qquad W \wedge B = B'$$

eşitlikleri mevcuttur. Darboux vektörü ile T teğet vektörü vektörel çarpıldığında

$$W \wedge T \;=\; zN - yB = T' \;\Rightarrow\; zN - yB = \kappa N$$

olur. Buradan $y = 0$ ve $z = \kappa$ bulunur. Darboux vektörü ile N asli normal vektörü vektörel çarpıldığında

$$W \wedge N \;=\; zT - yB = N' \;\Rightarrow\; -zT + xB = -\kappa T + \tau B$$

bulunur. Burada $z = \kappa$ olduğundan $x = \tau$ olur. Darboux vektörü ile B binormal vektörü vektörel çarpıldığında

$$W \wedge B \;=\; zT - yB = B' \;\Rightarrow\; yT - xN = -\tau N$$

olur. $y = 0$ olduğundan $x = \tau$ elde edilir. x, y, z değerleri (2.4.1) eşitliğinde yerine yazıldığında

$$W(s) \;=\; \tau(s)T(s) + \kappa(s)B(s)$$

bulunur. Birim Darboux vektörü $\overrightarrow{C}(s)$ olsun. Buradan

$$\overrightarrow{C}(s) = \frac{\tau(s)T(s) + \kappa(s)B(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}$$

olur. (Şekil 2.3)'den

$$\begin{aligned} cos\Theta &= \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \\ sin\Theta &= \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \end{aligned}$$

yazılır. Buradan birim Darboux vektörü

$$\overrightarrow{C}(s) = sin\Theta T + cos\Theta B$$

elde edilir.

3. BULGULAR ve TARTIŞMA

Bu bölüm çalışmanın orjinal kısmını oluşturmaktadır. Burada, diferensiyellenebilir herhangi bir α^* involüt eğrisinin Frenet vektörlerinin ve birim Darboux vektörünün vektörel moment vektörleri ve bu vektörlerin çizdiği eğriler tanımlandı. Bu eğrilerin Frenet vektörleri, eğrilikleri ve burulmaları hesaplandı. Son olarak elde edilen eğrilerin involüt eğrisiyle sabit genişlikli eğri kategorisinde olup olmadıkları araştırıldı.

3.1 İnvolüt Eğrisinin Frenet Vektörlerinden Elde Edilen Vektörel Moment Eğrileri

α eğrisinin involütü α^* ve Frenet vektörleri sırasıyla $\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\}$, $\{\overrightarrow{T}^*, \overrightarrow{N}^*, \overrightarrow{B}^*\}$ olsun. (2.1.4) bağıntısında (2.2.1) eşitliği yerine yazıldığında α^* involüt eğrisi

$$\alpha^* = (f+\lambda)\overrightarrow{T} + g\overrightarrow{N} + h\overrightarrow{B} \tag{3.1.1}$$

olur. Teorem (2.2.2) ifadesine benzer olarak Frenet vektörleri

$$\begin{aligned}
\overrightarrow{T} &= \frac{-\kappa\overrightarrow{N}^* + \tau\overrightarrow{B}^*}{\sqrt{\kappa^2+\tau^2}} \\
\overrightarrow{N} &= \overrightarrow{T}^* \\
\overrightarrow{B} &= \frac{\tau\overrightarrow{N}^* + \kappa\overrightarrow{B}^*}{\sqrt{\kappa^2+\tau^2}}
\end{aligned}$$

şeklinde yazılır. Bu eşitlikler (3.1.1) ifadesinde yerine yazıldığında

$$\alpha^* = g\overrightarrow{T}^* + \frac{h\tau - \kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\overrightarrow{N}^* + \frac{h\kappa + \tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\overrightarrow{B}^*$$

olur. α^* eğrisi Frenet vektörlerinin lineer birleşimi olarak

$$\alpha^* = f_1\overrightarrow{T}^* + g_1\overrightarrow{N}^* + h_1\overrightarrow{B}^*$$

biçiminde yazılır. Burada katsayılar

$$\begin{aligned} f_1 &= g, \\ g_1 &= \frac{h\tau - \kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}, \\ h_1 &= \frac{h\kappa + \tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}} \end{aligned} \tag{3.1.2}$$

şeklindedir. Bu katsayılar arasında (2.1.5) ifadesine benzer olarak

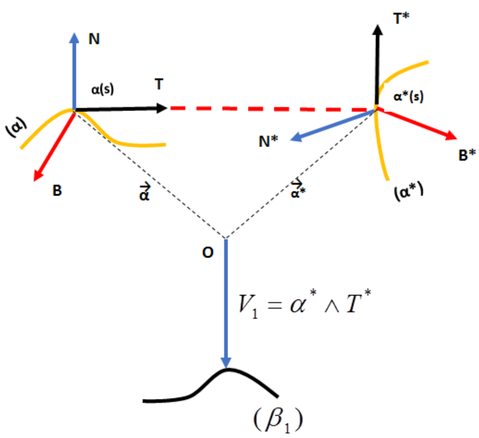
$$f_1' = 1 + g_1\kappa_1, \quad g_1' = h_1\tau_1 - f_1\kappa_1, \quad h_1' = -g_1\tau_1 \tag{3.1.3}$$

eşitlikleri vardır.

Tanım 3.1.1 İnvolüt eğrisinin $\overrightarrow{T}^*$ teğet vektörünün $\overrightarrow{V_1} = \alpha^* \wedge \overrightarrow{T}^*$ vektörel moment vektörünün çizdiği eğri β_1 ile gösterilsin. Bu eğri

$$\beta_1 = h_1\overrightarrow{N}^* - g_1\overrightarrow{B}^*$$

şeklindedir (Şekil 3.1).



Şekil 3.1: V_1 Vektörel Moment Vektörünün Çizdiği β_1 Eğrisi

Teorem 3.1.1 β_1 eğrisinin Frenet vektörleri $\overrightarrow{T}_{\beta_1}, \overrightarrow{N}_{\beta_1}, \overrightarrow{B}_{\beta_1}$ ile gösterilsin. Bu vektörlerin involüt eğrisinin Frenet vektörleri cinsinden karşılıkları

$$\overrightarrow{T}_{\beta_1} = \frac{-h_1\overrightarrow{T}^* + gB^*}{\sqrt{g^2+h_1^2}}$$

$$\overrightarrow{N}_{\beta_1} = \frac{\big(g_1(g\tau_1+h_1\kappa_1)+h_1\big)\big(g\overrightarrow{T}^*+h_1\overrightarrow{B}^*\big)-\big(g\tau_1+h_1\kappa_1\big)\big(g^2+h_1^2\big)\overrightarrow{N}^*}{\kappa_1^2\sqrt{\big((g^2+h_1^2)(g\tau_1+h_1\kappa_1)^2+(g_1(g\tau_1+h_1\kappa_1)+h_1)^2\big)\big(g^2+h_1^2\big)}}$$

$$\overrightarrow{B}_{\beta_1} = \frac{g\big(g\tau_1+h_1\kappa_1\big)\overrightarrow{T}^*+\big(g_1(g\tau_1+h_1\kappa_1)+h_1\big)\overrightarrow{N}^*+h_1\big(g\tau_1+h_1\kappa_1\big)\overrightarrow{B}^*}{\kappa_1^2\sqrt{(g^2+h_1^2)(g\tau_1+h_1\kappa_1)^2+(g_1(g\tau_1+h_1\kappa_1)+h_1)^2}}$$

bağıntılarıyla verilir.

İspat. β_1 eğrisinin türevi alınırsa

$$\begin{aligned}
\beta_1' &= h_1'\overrightarrow{N}^* + h_1\overrightarrow{N}^{*\prime} - g_1'\overrightarrow{B}^* - g_1\overrightarrow{B}^{*\prime} \\
&= h_1'\overrightarrow{N}^* + h_1(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) - g_1'\overrightarrow{B}^* - g_1(-\tau_1\overrightarrow{N}^*) \\
&= -h_1\kappa_1\overrightarrow{T}^* + (h_1' + g_1\tau_1)\overrightarrow{N}^* + (h_1\tau_1 - g_1')\overrightarrow{B}^* \\
&= -h_1\kappa_1\overrightarrow{T}^* + (\underbrace{-g_1\tau_1 + g_1\tau_1}_{0})\overrightarrow{N}^* + (\underbrace{h_1\tau_1 - (h_1\tau_1 - g\kappa_1)}_{g\kappa_1})\overrightarrow{B}^* \\
&= -h_1\kappa_1\overrightarrow{T}^* + g\kappa_1\overrightarrow{B}^* \qquad (3.1.4)
\end{aligned}$$

olur. Bu ifadenin normu

$$\| \beta_1' \| = \kappa_1\sqrt{g^2+h_1^2} \qquad (3.1.5)$$

olur.

(2.1.1) ifadesinde (3.1.4) ve (3.1.5) eşitlikleri yerlerine yazılırsa eğrinin teğet vektörü

$$\overrightarrow{T}_{\beta_1} = \frac{-h_1\overrightarrow{T}^* + g\overrightarrow{B}^*}{\sqrt{g^2+h_1^2}}$$

şeklinde olur. β_1' ifadesinin türevi alınırsa

$$\begin{aligned}
\beta_1'' &= -h_1'\kappa_1\overrightarrow{T}^* - h_1\kappa_1'\overrightarrow{T}^* - h_1\kappa_1\overrightarrow{T}^{*\prime} + g'\kappa_1\overrightarrow{B}^* + g\kappa_1'\overrightarrow{B}^* + g\kappa_1\overrightarrow{B}^{*\prime} \\
&= -h_1'\kappa_1\overrightarrow{T}^* - h_1\kappa_1'\overrightarrow{T}^* - h_1\kappa_1(\kappa_1\overrightarrow{N}^*) + g'\kappa_1\overrightarrow{B}^* + g\kappa_1'\overrightarrow{B}^* + g\kappa_1(-\tau_1\overrightarrow{N}^*) \\
&= -(h_1'\kappa_1 + h_1\kappa_1')\overrightarrow{T}^* - (g\kappa_1\tau_1 + h_1\kappa_1^2)\overrightarrow{N}^* + (g'\kappa_1 + g\kappa_1')\overrightarrow{B}^* \\
&= -(-(g_1\tau_1)\kappa_1 + h_1\kappa_1')\overrightarrow{T}^* - (g\kappa_1\tau_1 + h_1\kappa_1^2)\overrightarrow{N}^* + ((1+g_1\kappa_1)\kappa_1 + g\kappa_1')\overrightarrow{B}^* \\
&= (g_1\kappa_1\tau_1 - h_1\kappa_1')\overrightarrow{T}^* - (g\kappa_1\tau_1 + h_1\kappa_1^2)\overrightarrow{N}^* + (g\kappa_1' + g_1\kappa_1^2 + \kappa_1)\overrightarrow{B}^*
\end{aligned}$$

bulunur. β_1' ve β_1'' vektörel çarpıldığında

$$\beta_1' \wedge \beta_1'' = \kappa_1^2\Big(g(g\tau_1 + h_1\kappa_1)\overrightarrow{T}^* + (g_1(g\tau_1 + h_1\kappa_1) + h_1)\overrightarrow{N}^* + h_1(g\tau_1 + h_1\kappa_1)\overrightarrow{B}^*\Big)$$

eşitliği bulunur. Bu eşitliğin normu

$$\| \beta_1' \wedge \beta_1'' \| = \kappa_1^4\sqrt{(g^2+h_1^2)(g\tau_1 + h_1\kappa_1)^2 + (g_1(g\tau_1 + h_1\kappa_1) + h_1)^2}$$

olur. (2.1.1) ifadesinde $\beta_1' \wedge \beta_1''$ ve $\| \beta_1' \wedge \beta_1'' \|$ eşitlikleri kullanıldığında eğrinin binormal vektörü

$$\overrightarrow{B}_{\beta_1} = \frac{g(g\tau_1 + h_1\kappa_1)\overrightarrow{T}^* + (g_1(g\tau_1 + h_1\kappa_1) + h_1)\overrightarrow{N}^* + h_1(g\tau_1 + h_1\kappa_1)\overrightarrow{B}^*}{\kappa_1^2\sqrt{(g^2+h_1^2)(g\tau_1 + h_1\kappa_1)^2 + (g_1(g\tau_1 + h_1\kappa_1) + h_1)^2}}$$

bulunur.

(2.1.1) eşitliğinde $\overrightarrow{T}_{\beta_1}$ ve $\overrightarrow{B}_{\beta_1}$ vektörleri yerlerine yazılırsa eğrinin asli normal vekörü

$$\overrightarrow{N}_{\beta_1} = \frac{(g_1(g\tau_1+h_1\kappa_1)+h_1)(gT^*+h_1\overrightarrow{B}^*)-(g\tau_1+h_1\kappa_1)(g^2+h_1^2)\overrightarrow{N}^*}{\kappa_1^2\sqrt{\left((g^2+h_1^2)(g\tau_1+h_1\kappa_1)+(g_1(g\tau_1+h_1\kappa_1)+h_1)^2\right)\left(g^2+h_1^2\right)}}$$

elde edilir. □

Teorem 3.1.2 β_1 eğrisinin eğriliği ve torsiyonu sırasıyla κ_{β_1} ve τ_{β_1} ile gösterilsin. Bu eğriliklerin involüt eğrisinin eğrilikleri cinsinden karşılıkları

$$\kappa_{\beta_1} = \frac{\sqrt{(g^2+h_1^2)(g\tau_1+h_1\kappa_1)^2+(g_1(g\tau_1+h_1\kappa_1)+h_1)^2}}{\kappa_1(g^2+h_1^2)^{\frac{3}{2}}}$$

$$\tau_{\beta_1} = \frac{\begin{array}{c}\left(g\tau_1+h_1\kappa_1\right)\left(gg_1(2\kappa_1'\tau_1+\kappa_1\tau_1')+g_1h_1(\kappa_1\kappa_1'+\kappa_1^{2'})+2h_1\kappa_1'\right)\\ -\left(g_1(g\tau_1+h_1\kappa_1)+h_1\right)\left(2g\kappa_1'\tau_1+g\kappa_1\tau_1'+h_1\kappa_1\kappa_1'+h_1\kappa_1^{2'}+2\kappa_1\tau_1\right)\end{array}}{\kappa_1^2\Big((g^2+h_1^2)(g\tau_1+h_1\kappa_1)^2+(g_1(g\tau_1+h_1\kappa_1)+h_1)^2\Big)}$$

bağıntısıyla verilir.

İspat. (2.1.2) ifadesinde $\| \beta_1' \|$ ve $\| \beta_1' \wedge \beta_1'' \|$ eşitlikleri yerine yazıldığında eğrinin eğriliği

$$\kappa_{\beta_1} = \frac{\sqrt{(g^2+h_1^2)(g\tau_1+h_1\kappa_1)^2+(g_1(g\tau_1+h_1\kappa_1)+h_1)^2}}{\kappa_1(g^2+h_1^2)^{\frac{3}{2}}}$$

şeklinde olur. Eğrinin torsiyonunu hesaplamak için β_1'' ifadesinin türevi alınırsa

$$\begin{aligned}\beta_1''' &= (g_1'\kappa_1\tau_1+g_1\kappa_1'\tau_1+g_1\kappa_1\tau_1'-h_1'\kappa_1'-h_1\kappa_1'')\overrightarrow{T}^*+(g_1\kappa_1\tau_1-h_1\kappa_1')\overrightarrow{T}^{*'}\\ &-(h_1'\kappa_1^2+h_1\kappa_1^{2'}+g'\kappa_1\tau_1+g\kappa_1'\tau_1+g\kappa_1\tau_1')\overrightarrow{N}^*-(h_1\kappa_1^2+g\kappa_1\tau_1)\overrightarrow{N}^{*'}\\ &+(\kappa_1'+g_1'\kappa_1^2+g_1\kappa_1^{2'}+g'\kappa_1'+g\kappa_1'')\overrightarrow{B}^*+(\kappa_1+g_1\kappa_1^2+g\kappa_1')\overrightarrow{B}^{*'}\end{aligned}$$

$$
\begin{aligned}
&= (g_1'\kappa_1\tau_1 + g_1\kappa_1'\tau_1 + g_1\kappa_1\tau_1' - h_1'\kappa_1' - h_1\kappa_1'')\overrightarrow{T}^* + (g_1\kappa_1\tau_1 - h_1\kappa_1')(\kappa_1\overrightarrow{N}^*) \\
&\quad -(h_1'\kappa_1^2 + h_1\kappa_1^{2'} + g'\kappa_1\tau_1 + g\kappa_1'\tau_1 + g\kappa_1\tau_1')\overrightarrow{N}^* - (h_1\kappa_1^2 + g\kappa_1\tau_1)(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) \\
&\quad +(\kappa_1' + g_1'\kappa_1^2 + g_1\kappa_1^{2'} + g'\kappa_1' + g\kappa_1'')\overrightarrow{B}^* + (\kappa_1 + g_1\kappa_1^2 + g\kappa_1')(-\tau_1\overrightarrow{N}^*) \\
&= (g_1'\kappa_1\tau_1 + g_1\kappa_1'\tau_1 + g_1\kappa_1\tau_1' - h_1'\kappa_1' - h_1\kappa_1'' + h_1\kappa_1^3 + g\kappa_1^2\tau_1)\overrightarrow{T}^* \\
&\quad +(-h_1'\kappa_1^2 - h_1\kappa_1^{2'} - g'\kappa_1\tau_1 - g\kappa_1'\tau_1 - g\kappa_1\tau_1' + g_1\kappa_1^2\tau_1 - h_1\kappa_1\kappa_1' - \kappa_1\tau_1 - g_1\kappa_1^2\tau_1 \\
&\quad -g\kappa_1'\tau_1)\overrightarrow{N}^* + (-h_1\kappa_1^2\tau_1 - g\kappa_1\tau_1^2 + \kappa_1' + g_1'\kappa_1^2 + g_1\kappa_1^{2'} + g'\kappa_1' + g\kappa_1'')\overrightarrow{B}^* \\
&= (2g_1\kappa_1'\tau_1 + g_1\kappa_1\tau_1' + h_1\kappa_1\tau_1^2 + h_1\kappa_1^3 - h_1\kappa_1'')\overrightarrow{T}^* - (2g\kappa_1'\tau_1 + g\kappa_1\tau_1' + h_1\kappa_1\kappa_1' \\
&\quad +h_1\kappa_1^{2'} + 2\kappa_1\tau_1)\overrightarrow{N}^* + (g\kappa_1'' - g\kappa_1^3 - g\kappa_1\tau_1^2 + g_1\kappa_1\kappa_1' + g_1\kappa_1^{2'} + 2\kappa_1')\overrightarrow{B}^*
\end{aligned}
$$

olur. β_1', β_1'' ve β_1''' ifadelerinin determinantı alınırsa

$$
\begin{aligned}
\det(\beta_1', \beta_1'', \beta_1''') &= \kappa_1^2\Big(\big(g\tau_1 + h_1\kappa_1\big)\big(gg_1(2\kappa_1'\tau_1 + \kappa_1\tau_1') + g_1h_1(\kappa_1\kappa_1' + \kappa_1^{2'}) + 2h_1\kappa_1'\big) \\
&\quad -\big(g_1(g\tau_1 + h_1\kappa_1) + h_1\big)\big(2g\kappa_1'\tau_1 + g\kappa_1\tau_1' + h_1\kappa_1\kappa_1' + h_1\kappa_1^{2'} + 2\kappa_1\tau_1\big)\Big)
\end{aligned}
$$

bulunur. (2.1.2) ifadesinde $\| \beta_1' \wedge \beta_1'' \|$ ve $\det(\beta_1', \beta_1'', \beta_1''')$ eşitlikleri yerlerine yazıldığında eğrinin torsiyonu

$$
\tau_{\beta_1} = \frac{\begin{matrix}\big(g\tau_1 + h_1\kappa_1\big)\big(gg_1(2\kappa_1'\tau_1 + \kappa_1\tau_1') + g_1h_1(\kappa_1\kappa_1' + \kappa_1^{2'}) + 2h_1\kappa_1'\big) \\ -\big(g_1(g\tau_1 + h_1\kappa_1) + h_1\big)\big(2g\kappa_1'\tau_1 + g\kappa_1\tau_1' + h_1\kappa_1\kappa_1' + h_1\kappa_1^{2'} + 2\kappa_1\tau_1\big)\end{matrix}}{\kappa_1^2\Big((g^2 + h_1^2)(g\tau_1 + h_1\kappa_1)^2 + (g_1(g\tau_1 + h_1\kappa_1) + h_1)^2\Big)}
$$

elde edilir. □

Sonuç 3.1.1 β_1 eğrisinin $\overrightarrow{T}_{\beta_1}, \overrightarrow{N}_{\beta_1}, \overrightarrow{B}_{\beta_1}$ Frenet vektörlerinin α eğrisinin Frenet vektörleri cinsinden eşitleri

$$\overrightarrow{T}_{\beta_1} = \frac{g\tau\overrightarrow{T} - (h\kappa + \tau(f+\lambda))\overrightarrow{N} + g\kappa\overrightarrow{B}}{\sqrt{g^2(\kappa^2+\tau^2) + (h\kappa + \tau(f+\lambda))^2}}$$

$$\overrightarrow{N}_{\beta_1} = \frac{\begin{gathered}\Big(\frac{h^2\kappa\tau^2 + \tau(f+\lambda)(h(\tau^2-\kappa^2) - \kappa\tau(f+\lambda))}{\kappa^2+\tau^2} + \frac{g^2\kappa(\kappa^2+\tau^2) + \kappa(h\kappa+\tau(f+\lambda))^2}{(\sqrt{\kappa^2+\tau^2})^3} \\ + \frac{\lambda\kappa\tau(h\kappa+\tau(f+\lambda))^2}{\sqrt{\kappa^2+\tau^2}(g((\frac{\tau}{\kappa})'\kappa^2) + (h\kappa+\tau(f+\lambda))(\kappa^2+\tau^2))}\Big)\overrightarrow{T} \\ g\Big(\frac{h\tau - \kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}} + \frac{\lambda\kappa(h\kappa+\tau(f+\lambda))\sqrt{\kappa^2+\tau^2}}{g((\frac{\tau}{\kappa})'\kappa^2) + (h\kappa+\tau(f+\lambda))(\kappa^2+\tau^2)}\Big)\overrightarrow{N} \\ +\Big(\frac{h^2\kappa^2\tau + \kappa(f+\lambda)(h(\tau^2-\kappa^2) - \kappa\tau(f+\lambda))}{\kappa^2+\tau^2} - \frac{g^2\tau(\kappa^2+\tau^2) + \tau(h\kappa+\tau(f+\lambda))^2}{(\sqrt{\kappa^2+\tau^2})^3} \\ + \frac{\lambda\kappa^2(h\kappa+\tau(f+\lambda))^2}{\sqrt{\kappa^2+\tau^2}(g((\frac{\tau}{\kappa})'\kappa^2) + (h\kappa+\tau(f+\lambda))(\kappa^2+\tau^2))}\Big)\overrightarrow{B}\end{gathered}}{\left(\begin{gathered}\Bigg(\Big(\frac{g^2(\kappa^2+\tau^2) + (h\kappa+\tau(f+\lambda))^2}{\kappa^2+\tau^2}\Big)\Big(\frac{g\kappa^2(\frac{\tau}{\kappa})' + (\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))}{\lambda\kappa(\kappa^2+\tau^2)}\Big)^2 \\ +\Big(\frac{h\tau - \kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{g\kappa^2(\frac{\tau}{\kappa})' + (\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))}{\lambda\kappa(\kappa^2+\tau^2)}\big) \\ + \frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big)^2\Bigg)\Bigg(\frac{g^2(\kappa^2+\tau^2) + (h\kappa+\tau(f+\lambda))^2}{\kappa^2+\tau^2}\Bigg)\end{gathered}\right)^{\frac{1}{2}}}$$

$$(f+\lambda)\Big(g(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))\Big)-\lambda\kappa^2(h\kappa+\tau(f+\lambda))\overrightarrow{T}$$

$$+g^2(\frac{\tau}{\kappa})'+g\big(\kappa^2+\tau^2\big)\big(h\kappa+\tau(f+\lambda)\big)\overrightarrow{N}$$

$$\overrightarrow{B}_{\beta_1} = \frac{h\Big(g(\frac{\tau}{\kappa})'+\big(\kappa^2+\tau^2\big)\big(h\kappa+\tau(f+\lambda)\big)\Big)+\lambda\kappa(h\kappa\tau+\tau^2(f+\lambda))\overrightarrow{B}}{\frac{\sqrt{\kappa^2+\tau^2}}{\lambda^2\kappa^2}\Bigg(\Big(g^2(\kappa^2+\tau^2)+(h\kappa+\tau(f+\lambda))^2\Big)\Big(g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))\Big)^2}$$

$$+\Big(\big(h\tau-\kappa(f+\lambda)\big)\big(g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))\big)$$

$$+\lambda\kappa(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))\Big)^2\Bigg)^{\frac{1}{2}}$$

bağıntılarıyla verilir.

Sonuç 3.1.2 β_1 eğrisinin κ_{β_1} ve τ_{β_1} eğriliklerinin α esas eğrinin eğrilikleri cinsinden yazılışı

$$\kappa_{\beta_1} = \frac{\Bigg(\Big(\frac{g^2(\kappa^2+\tau^2)+(h\kappa+\tau(f+\lambda))^2}{\kappa^2+\tau^2}\Big)\Big(\frac{g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))}{\lambda\kappa(\kappa^2+\tau^2)}\Big)^2 +\Big(\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))}{\lambda\kappa(\kappa^2+\tau^2)}\big)+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big)^2\Bigg)^{\frac{1}{2}}}{\big(\frac{g^2(\kappa^2+\tau^2)+(h\kappa+\tau(f+\lambda))^2}{\lambda\kappa(\kappa^2+\tau^2)}\big)^{\frac{3}{2}}}$$

$$\tau_{\beta_1} = \frac{X_1-Y_1+Z_1}{\Big(\frac{g^2(\kappa^2+\tau^2)+(h\kappa+\tau(f+\lambda))^2}{\kappa^2+\tau^2}\Big)\Big(\frac{g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))}{\lambda\kappa(\kappa^2+\tau^2)}\Big)^2 +\Big(\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))}{\lambda\kappa(\kappa^2+\tau^2)}\big)+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big)^2}$$

şeklinde verilir. Burada X_1, Y_1 ve Z_1

$$X_1 = g\Big(\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}(\frac{2\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'+(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})$$

$$+\frac{h\kappa+\tau(f+\lambda)}{\lambda\kappa}(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})^2)-\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''\Big)$$

$$Y_1 = \Big((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{2g\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}+\frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+\frac{h\kappa+\tau(f+\lambda)}{\lambda\kappa}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'$$

$$+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2})'+\frac{2(\frac{\tau}{\kappa})'}{\lambda^2\sqrt{\kappa^2+\tau^2}}\Big)\Big(\frac{\lambda\kappa(h\kappa+\tau(f+\lambda))\sqrt{\kappa^2+\tau^2}}{g\kappa^2(\frac{\tau}{\kappa})'+(h\kappa+\tau(f+\lambda))(\kappa^2+\tau^2)}$$

$$+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big)$$

$$Z_1 = \Big(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big)\Big((\frac{2\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'+g(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''-\frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2})$$

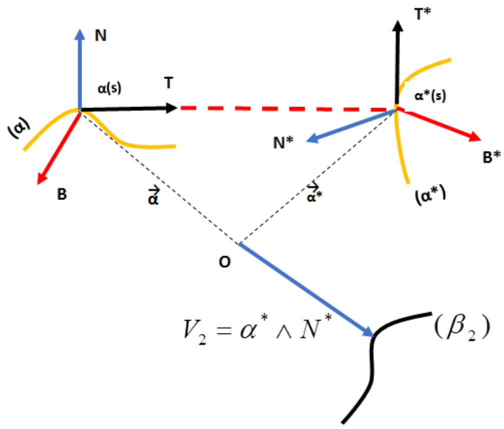
$$+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'+(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2})')\Big)$$

şeklinde birer katsayıdır.

Tanım 3.1.2 İnvolüt eğrisinin $\overrightarrow{N}^*$ asli normal vektörünün $\overrightarrow{V_2}=\alpha^*\wedge\overrightarrow{N}^*$ vektörel moment vektörünün çizdiği eğri β_2 ile gösterilsin. Bu eğri

$$\beta_2 = -h_1\overrightarrow{T}^* + g\overrightarrow{B}^*$$

şeklindedir (Şekil 3.2).



Şekil 3.2: V_2 Vektörel Moment Vektörünün Çizdiği β_2 Eğrisi

Teorem 3.1.3 β_2 eğrisinin Frenet vektörleri $\overrightarrow{T}_{\beta_2}, \overrightarrow{N}_{\beta_2}, \overrightarrow{B}_{\beta_2}$ ile gösterilsin. Bu vektörlerin involüt eğrisinin Frenet vektörleri cinsinden karşılıkları

$$\overrightarrow{T}_{\beta_2} = \frac{g_1\tau_1\overrightarrow{T}^* - (h_1\kappa_1 + g\tau_1)\overrightarrow{N}^* + (1+g_1\kappa_1)\overrightarrow{B}^*}{\sqrt{g_1^2\tau_1^2 + (h_1\kappa_1+g\tau_1)^2 + (1+g_1\kappa_1)^2}}$$

$$\overrightarrow{N}_{\beta_2} = \frac{\begin{array}{c}\Big(Y_2(1+g_1\kappa_1) + Z_2(g\tau_1 + h_1\kappa_1)\Big)\overrightarrow{T}^* + \Big(Z_2g_1\tau_1 - X_2(1+g_1\kappa_1)\Big)\overrightarrow{N}^* \\ -\Big(X_2h_1\kappa_1 + X_2g\tau_1 + Y_2g_1\tau_1\Big)\overrightarrow{B}^*\end{array}}{\sqrt{\Big(g_1^2\tau_1^2 + (h_1\kappa_1+g\tau_1)^2 + (1+g_1\kappa_1)^2\Big)\Big({X_2}^2 + {Y_2}^2 + {Z_2}^2\Big)}}$$

$$\overrightarrow{B}_{\beta_2} = \frac{X_2\overrightarrow{T}^* + Y_2\overrightarrow{N}^* + Z_2\overrightarrow{B}^*}{\sqrt{{X_2}^2 + {Y_2}^2 + {Z_2}^2}}$$

şeklinde verilir. Burada X_2, Y_2 ve Z_2

$$X_2 = gh_1\kappa_1\tau_1^2 - gg_1\kappa_1'\tau_1 + g^2\tau_1^3 + g\tau_1' + gg_1\kappa_1\tau_1' + gh_1\kappa_1^3 + g^2\kappa_1^2\tau_1 + 2g_1\kappa_1\tau_1 + h_1\kappa_1' + 2\tau_1$$

$$Y_2 = gg_1\tau_1^3 + gg_1\kappa_1^2\tau_1 + g_1\tau_1' + g_1^2\kappa_1\tau_1' + g_1h_1\kappa_1^3 + g_1h_1\kappa_1\tau_1^2 - g_1^2\kappa_1'\tau_1 + h_1\kappa_1^2 + h_1\tau_1^2$$

$$Z_2 = gh_1\tau_1^3 + gh_1\kappa_1^2\tau_1 + g_1h_1\kappa_1\tau_1' - 2g_1\tau_1^2 - g_1h_1\kappa_1'\tau_1 + h_1^2\kappa_1^3 + h_1^2\kappa_1\tau_1^2$$

biçiminde birer katsayıdır.

İspat. β_2 eğrisinin türevi alınırsa

$$\begin{aligned}
\beta_2' &= -h_1'\overrightarrow{T}^* - h_1\overrightarrow{T}^{*'} + g'\overrightarrow{B}^* + g\overrightarrow{B}^{*'} \\
&= -h_1'\overrightarrow{T}^* - h_1\kappa_1\overrightarrow{N}^* + g'\overrightarrow{B}^* + g(-\tau_1\overrightarrow{N}^*) \\
&= -h_1'\overrightarrow{T}^* - (g\tau_1 + h_1\kappa_1)\overrightarrow{N}^* + g'\overrightarrow{B}^* \\
&= g_1\tau_1\overrightarrow{T}^* - (g\tau_1 + h_1\kappa_1)\overrightarrow{N}^* + (1 + g_1\kappa_1)\overrightarrow{B}^* \qquad (3.1.6)
\end{aligned}$$

şeklinde olur. Buradan norm alındığında

$$\| \beta_2' \| = \sqrt{g_1^2\tau_1^2 + (g\tau_1 + h_1\kappa_1)^2 + (1 + g_1\kappa_1)^2} \qquad (3.1.7)$$

ifadesi bulunur. (3.1.6) ve (3.1.7) bağıntılarından $\overrightarrow{T}_{\beta_2}$ vektörü

$$\overrightarrow{T}_{\beta_2} = \frac{g_1\tau_1\overrightarrow{T}^* - (g\tau_1 + h_1\kappa_1)\overrightarrow{N}^* + (1 + g_1\kappa_1)\overrightarrow{B}^*}{\sqrt{g_1^2\tau_1^2 + (g\tau_1 + h_1\kappa_1)^2 + (1 + g_1\kappa_1)^2}}$$

olur. β_2' ifadesinin türevi alınırsa

$$\begin{aligned}
\beta_2'' = {} & g_1'\tau_1\overrightarrow{T}^* + g_1\tau_1'\overrightarrow{T}^* + g_1\tau_1\overrightarrow{T}^{*'} - (h_1'\kappa_1 + h_1\kappa_1' + f_1'\tau_1 + g\tau_1')\overrightarrow{N}^* - (g\tau_1 + h_1\kappa_1)\overrightarrow{N}^{*'} \\
& + (g_1'\kappa_1 + g_1\kappa_1')\overrightarrow{B}^* + (1 + g_1\kappa_1)\overrightarrow{B}^{*'}
\end{aligned}$$

$$
\begin{aligned}
&= g_1'\tau_1\overrightarrow{T}^* + g_1\tau_1'\overrightarrow{T}^* + g_1\tau_1(\kappa_1\overrightarrow{N}^*) - (h_1'\kappa_1 + h_1\kappa_1' + f_1'\tau_1 + g\tau_1')\overrightarrow{N}^* \\
&\quad -(g\tau_1 + h_1\kappa_1)(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) + (g_1'\kappa_1 + g_1\kappa_1')\overrightarrow{B}^* + (1 + g_1\kappa_1)(-\tau_1\overrightarrow{N}^*) \\
&= (g_1'\tau_1 + g_1\tau_1' + h_1\kappa_1^2 + g\kappa_1\tau_1)\overrightarrow{T}^* + (-h_1'\kappa_1 - h_1\kappa_1' - f_1'\tau_1 - g\tau_1' - \tau_1)\overrightarrow{N}^* \\
&\quad +(-h_1\kappa_1\tau_1 - g\tau_1^2 + g_1'\kappa_1 + g_1\kappa_1')\overrightarrow{B}^* \\
&= ((h_1\tau_1 - g\kappa_1)\tau_1 + g_1\tau_1' + h_1\kappa_1^2 + g\kappa_1\tau_1)\overrightarrow{T}^* + (-(-g_1\tau_1)\kappa_1 - h_1\kappa_1' - (1 + g_1\kappa_1)\tau_1 \\
&\quad -g\tau_1' - \tau_1)\overrightarrow{N}^* + (-h_1\kappa_1\tau_1 - g\tau_1^2 + (h_1\tau_1 - g\kappa_1)\kappa_1 + g_1\kappa_1')\overrightarrow{B}^* \\
&= (g_1\tau_1' + h_1\kappa_1^2 + h_1\tau_1^2)\overrightarrow{T}^* - (g\tau_1' + 2\tau_1 + h_1\kappa_1')\overrightarrow{N}^* + (g_1\kappa_1' - g\kappa_1^2 - g\tau_1^2)\overrightarrow{B}^*
\end{aligned}
$$

eşitliği bulunur. β_2' ve β_2'' ifadeleri vektörel çarpıldığında

$$
\begin{aligned}
\beta_2' \wedge \beta_2'' &= \Big(gh_1\kappa_1\tau_1^2 - gg_1\kappa_1'\tau_1 + g^2\tau_1^3 + g\tau_1' + gg_1\kappa_1\tau_1' + gh_1\kappa_1^3 + g^2\kappa_1^2\tau_1 + 2g_1\kappa_1\tau_1 \\
&\quad +h_1\kappa_1' + 2\tau_1\Big)\overrightarrow{T}^* + \Big(gg_1\tau_1^3 + gg_1\kappa_1^2\tau_1 + g_1\tau_1' + g_1^2\kappa_1\tau_1' + g_1h_1\kappa_1^3 + g_1h_1\kappa_1\tau_1^2 \\
&\quad -g_1^2\kappa_1'\tau_1 + h_1\kappa_1^2 + h_1\tau_1^2\Big)\overrightarrow{N}^* + \Big(gh_1\tau_1^3 + gh_1\kappa_1^2\tau_1 + g_1h_1\kappa_1\tau_1' - 2g_1\tau_1^2 \\
&\quad -g_1h_1\kappa_1'\tau_1 + h_1^2\kappa_1^3 + h_1^2\kappa_1\tau_1^2\Big)\overrightarrow{B}^* \qquad (3.1.8)
\end{aligned}
$$

olur. Bu ifadenin normu

$$\| \beta_2' \wedge \beta_2'' \| = \left(\begin{array}{l} \big(gh_1\kappa_1\tau_1^2 - gg_1\kappa_1'\tau_1 + g^2\tau_1^3 + g\tau_1' + gg_1\kappa_1\tau_1' + gh_1\kappa_1^3 + g^2\kappa_1^2\tau_1 + 2g_1\kappa_1\tau_1 \\ +h_1\kappa_1' + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 + gg_1\kappa_1^2\tau_1 + g_1\tau_1' + g_1^2\kappa_1\tau_1' + g_1h_1\kappa_1^3 + g_1h_1\kappa_1\tau_1^2 \\ -g_1^2\kappa_1'\tau_1 + h_1\kappa_1^2 + h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 + gh_1\kappa_1^2\tau_1 + g_1h_1\kappa_1\tau_1' - g_1h_1\kappa_1'\tau_1 \\ +h_1^2\kappa_1^3 + h_1^2\kappa_1\tau_1^2 - 2g_1\tau_1^2\big)^2 \end{array} \right)^{\frac{1}{2}} \tag{3.1.9}$$

şeklide olur. (2.1.1) ifadesinde (3.1.8) ve (3.1.9) eşitlikleri yerlerine yazıldığında $\overrightarrow{B}_{\beta_2}$ vektörü

$$\overrightarrow{B}_{\beta_2} = \frac{\begin{array}{l} \Big(gh_1\kappa_1\tau_1^2 - gg_1\kappa_1'\tau_1 + g^2\tau_1^3 + g\tau_1' + gg_1\kappa_1\tau_1' + gh_1\kappa_1^3 + g^2\kappa_1^2\tau_1 \\ +2g_1\kappa_1\tau_1 + h_1\kappa_1' + 2\tau_1\Big)\overrightarrow{T}^* + \Big(gg_1\tau_1^3 + gg_1\kappa_1^2\tau_1 + g_1\tau_1' + g_1^2\kappa_1\tau_1' + g_1h_1\kappa_1^3 \\ +g_1h_1\kappa_1\tau_1^2 - g_1^2\kappa_1'\tau_1 + h_1\kappa_1^2 + h_1\tau_1^2\Big)\overrightarrow{N}^* + \Big(gh_1\tau_1^3 + gh_1\kappa_1^2\tau_1 \\ +g_1h_1\kappa_1\tau_1' + 2g_1\tau_1^3 - g_1h_1\kappa_1'\tau_1 + h_1^2\kappa_1^3 + h_1^2\kappa_1\tau_1^2\Big)\overrightarrow{B}^* \end{array}}{\begin{array}{l} \Big(\big(gh_1\kappa_1\tau_1^2 - gg_1\kappa_1'\tau_1 + g^2\tau_1^3 + g\tau_1' + gg_1\kappa_1\tau_1' + gh_1\kappa_1^3 + g^2\kappa_1^2\tau_1 \\ +2g_1\kappa_1\tau_1 + h_1\kappa_1' + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 + gg_1\kappa_1^2\tau_1 + g_1\tau_1' + g_1^2\kappa_1\tau_1' + g_1h_1\kappa_1^3 \\ +g_1h_1\kappa_1\tau_1^2 - g_1^2\kappa_1'\tau_1 + h_1\kappa_1^2 + h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 + gh_1\kappa_1^2\tau_1 \\ +g_1h_1\kappa_1\tau_1' - 2g_1\tau_1^2 - g_1h_1\kappa_1'\tau_1 + h_1^2\kappa_1^3 + h_1^2\kappa_1\tau_1^2\big)^2\Big)^{\dfrac{1}{2}} \end{array}}$$

bulunur. Burada katsayılar

$$X_2 = gh_1\kappa_1\tau_1^2 - gg_1\kappa_1'\tau_1 + g^2\tau_1^3 + g\tau_1' + gg_1\kappa_1\tau_1' + gh_1\kappa_1^3 + g^2\kappa_1^2\tau_1 + 2g_1\kappa_1\tau_1 + h_1\kappa_1' + 2\tau_1$$

$$Y_2 = gg_1\tau_1^3 + gg_1\kappa_1^2\tau_1 + g_1\tau_1' + g_1^2\kappa_1\tau_1' + g_1h_1\kappa_1^3 + g_1h_1\kappa_1\tau_1^2 - g_1^2\kappa_1'\tau_1 + h_1\kappa_1^2 + h_1\tau_1^2$$

$$Z_2 = gh_1\tau_1^3 + gh_1\kappa_1^2\tau_1 + g_1h_1\kappa_1\tau_1' - 2g_1\tau_1^2 - g_1h_1\kappa_1'\tau_1 + h_1^2\kappa_1^3 + h_1^2\kappa_1\tau_1^2$$

alınırsa $\overrightarrow{B}_{\beta_2}$ vektörü

$$\overrightarrow{B}_{\beta_2} = \frac{X_2\overrightarrow{T}^* + Y_2\overrightarrow{N}^* + Z_2\overrightarrow{B}^*}{\sqrt{{X_2}^2 + {Y_2}^2 + {Z_2}^2}}$$

olur. (2.1.1) ifadesinde $\overrightarrow{T}_{\beta_2}$ ve $\overrightarrow{B}_{\beta_2}$ vektörleri yerlerine yazıldığında eğrinin asli normal vektörü

$$\overrightarrow{N}_{\beta_2} = \frac{\begin{array}{c}\Big(Y_2(1+g_1\kappa_1) + Z_2(g\tau_1 + h_1\kappa_1)\Big)\overrightarrow{T}^* + \Big(Z_2g_1\tau_1 - X_2(1+g_1\kappa_1)\Big)\overrightarrow{N}^* \\ -\Big(X_2h_1\kappa_1 + X_2g\tau_1 + Y_2g_1\tau_1\Big)\overrightarrow{B}^*\end{array}}{\sqrt{\Big(g_1^2\tau_1^2 + (h_1\kappa_1 + g\tau_1)^2 + (1+g_1\kappa_1)^2\Big)\Big({X_2}^2 + {Y_2}^2 + {Z_2}^2\Big)}}$$

elde edilir. □

Teorem 3.1.4 β_2 eğrisinin eğriliği ve torsiyonu sırasıyla κ_{β_2} ve τ_{β_2} ile gösterilsin. Bu eğriliklerin involüt eğrisinin eğrilikleri cinsinden karşılıkları

$$\kappa_{\beta_2} = \frac{\sqrt{{X_2}^2 + {Y_2}^2 + {Z_2}^2}}{\Big(g_1^2\tau_1^2 + (h_1\kappa_1 + g\tau_1)^2 + (1 + g_1\kappa_1)^2\Big)^{\frac{3}{2}}}$$

$$\tau_{\beta_2} = \frac{\begin{array}{c}\Big(g_1\tau_1' + h_1\kappa_1^2 + h_1\tau_1^2\Big)\Big(Y_3(1 + g_1\kappa_1) + Z_3(g\tau_1 + h_1\kappa_1)\Big) - \Big(g_1\kappa_1' - g\kappa_1^2 - g\tau_1^2\Big). \\ \Big(Y_3g_1\tau_1 - X_3(g\tau_1 + h_1\kappa_1)\Big) + \Big(g\tau_1' + 2\tau_1 + h_1\kappa_1'\Big)\Big(X_3(1 + g_1\kappa_1) - Z_3g_1\tau_1\Big)\end{array}}{{X_2}^2 + {Y_2}^2 + {Z_2}^2}$$

bağıntılarıyla verilir. Burada X_2, Y_2, Z_2, X_3, Y_3 ve Z_3

$$X_2 = gh_1\kappa_1\tau_1^2 - gg_1\kappa_1'\tau_1 + g^2\tau_1^3 + g\tau_1' + gg_1\kappa_1\tau_1' + gh_1\kappa_1^3 + g^2\kappa_1^2\tau_1 + 2g_1\kappa_1\tau_1 + h_1\kappa_1' + 2\tau_1$$

$$Y_2 = gg_1\tau_1^3 + gg_1\kappa_1^2\tau_1 + g_1\tau_1' + g_1^2\kappa_1\tau_1' + g_1h_1\kappa_1^3 + g_1h_1\kappa_1\tau_1^2 - g_1^2\kappa_1'\tau_1 + h_1\kappa_1^2 + h_1\tau_1^2$$

$$Z_2 = gh_1\tau_1^3 + gh_1\kappa_1^2\tau_1 + g_1h_1\kappa_1\tau_1' - 2g_1\tau_1^2 - g_1h_1\kappa_1'\tau_1 + h_1^2\kappa_1^3 + h_1^2\kappa_1\tau_1^2$$

$$X_3 = g_1\tau_1'' - g_1\kappa_1^2\tau_1 - g_1\tau_1^3 + h_1\tau_1\tau_1' + h_1\kappa_1^{2'} + h_1\tau_1^{2'} + h_1\kappa_1\kappa_1' + 2\kappa_1\tau_1$$

$$Y_3 = g\tau_1^3 + g\kappa_1^2\tau_1 - g\tau_1'' + h_1\kappa_1^3 + h_1\kappa_1\tau_1^2 - h_1\kappa_1'' - 3\tau_1'$$

$$Z_3 = g_1\kappa_1'' - g\kappa_1^{2'} - g\kappa_1\kappa_1' - g\tau_1^{2'} - g\tau_1\tau_1' - g_1\kappa_1^3 - g_1\kappa_1\tau_1^2 - \kappa_1^2 - 3\tau_1^2$$

şeklinde birer katsayıdır.

İspat. (2.1.2) ifadesinde (3.1.7) ve (3.1.9) eşitlikleri yerine yazıldığında eğrinin eğriliği

$$\kappa_{\beta_2} = \frac{\sqrt{{X_2}^2 + {Y_2}^2 + {Z_2}^2}}{\left(g_1^2\tau_1^2 + (h_1\kappa_1 + g\tau_1)^2 + (1 + g_1\kappa_1)^2\right)^{\frac{3}{2}}}$$

olur. Torsiyonu bulmak için β_2'' ifadesinin türevi alınırsa

$$\begin{aligned}
\beta''' &= (g_1'\tau_1' + g_1\tau_1'' + h_1'\kappa_1^2 + h_1\kappa_1^{2'} + h_1'\tau_1^2 + h_1\tau_1^{2'})\overrightarrow{T}^* + (g_1\tau_1' + h_1\kappa_1^2 + h_1\tau_1^2)\overrightarrow{T}^{*'} \\
&\quad -(g'\tau_1' + g\tau_1'' + 2\tau_1' + h_1'\kappa_1' + h_1\kappa_1'')\overrightarrow{N}^* - (g\tau_1' + 2\tau_1 + h_1\kappa_1')\overrightarrow{N}^{*'} + (g_1'\kappa_1' + g_1\kappa_1'' \\
&\quad -g'\tau_1^2 - g\tau_1^{2'} - g'\kappa_1^2 - g\kappa_1^{2'})\overrightarrow{B}^* + (g_1\kappa_1' - g\tau_1^2 - g\kappa_1^2)\overrightarrow{B}^{*'} \\
&= (g_1'\tau_1' + g_1\tau_1'' + h_1'\kappa_1^2 + h_1\kappa_1^{2'} + h_1'\tau_1^2 + h_1\tau_1^{2'})\overrightarrow{T}^* + (g_1\tau_1' + h_1\kappa_1^2 + h_1\tau_1^2)(\kappa_1\overrightarrow{N}^*) \\
&\quad -(g'\tau_1' + g\tau_1'' + 2\tau_1' + h_1'\kappa_1' + h_1\kappa_1'')\overrightarrow{N}^* - (g\tau_1' + 2\tau_1 + h_1\kappa_1')(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) \\
&\quad +(g_1'\kappa_1' + g_1\kappa_1'' - g'\tau_1^2 - g\tau_1^{2'} - g'\kappa_1^2 - g\kappa_1^{2'})\overrightarrow{B}^* + (g_1\kappa_1' - g\tau_1^2 - g\kappa_1^2)(-\tau_1\overrightarrow{N}^*) \\
&= (g_1'\tau_1' + g_1\tau_1'' + h_1'\kappa_1^2 + h_1\kappa_1^{2'} + h_1'\tau_1^2 + h_1\tau_1^{2'} + g\kappa_1\tau_1' + 2\kappa_1\tau_1 + h_1\kappa_1\kappa_1')\overrightarrow{T}^* \\
&\quad +(g_1\kappa_1\tau_1' + h_1\kappa_1^3 + h_1\kappa_1\tau_1^2 - g'\tau_1' - g\tau_1'' - 2\tau_1' - h_1'\kappa_1' - h_1\kappa_1'' - g_1\kappa_1'\tau_1 + g\tau_1^3 \\
&\quad +g\kappa_1^2\tau_1)\overrightarrow{N}^* + (g_1'\kappa_1' + g_1\kappa_1'' - g'\tau_1^2 - g\tau_1^{2'} - g\tau_1\tau_1' - 2\tau_1^2 - h_1\kappa_1'\tau_1 - g'\kappa_1^2 - g\kappa_1^{2'})\overrightarrow{B}^*
\end{aligned}$$

$$
\begin{aligned}
&= \Big((h_1\tau_1 - g\kappa_1)\tau_1' + g_1\tau_1'' + (-g_1\tau_1)\kappa_1^2 + h_1\kappa_1^{2'} + (-g_1\tau_1)\tau_1^2 + h_1\tau_1^{2'} + g\kappa_1\tau_1' \\
&\quad +2\kappa_1\tau_1 + h_1\kappa_1\kappa_1'\Big)\overrightarrow{T}^* + \Big(g_1\kappa_1\tau_1' + h_1\kappa_1^3 + h_1\kappa_1\tau_1^2 - (1+g_1\kappa_1)\tau_1' - g\tau_1'' - 2\tau_1' \\
&\quad -(-g_1\tau_1)\kappa_1' - h_1\kappa_1'' - g_1\kappa_1'\tau_1 + g\tau_1^3 + g\kappa_1^2\tau_1\Big)\overrightarrow{N}^* + \Big((h_1\tau_1 - g\kappa_1)\kappa_1' + g_1\kappa_1'' \\
&\quad -(1+g_1\kappa_1)\tau_1^2 - g\tau_1^{2'} - g\tau_1\tau_1' - 2\tau_1^2 - h_1\kappa_1'\tau_1 - (1+g_1\kappa_1)\kappa_1^2 - g\kappa_1^{2'}\Big) \\
&= \Big(g_1\tau_1'' - g_1\kappa_1^2\tau_1 - g_1\tau_1^3 + h_1\tau_1\tau_1' + h_1\kappa_1^{2'} + h_1\tau_1^{2'} + h_1\kappa_1\kappa_1' + 2\kappa_1\tau_1\Big)\overrightarrow{T}^* \\
&\quad +\Big(g\tau_1^3 + g\kappa_1^2\tau_1 - g\tau_1'' + h_1\kappa_1^3 + h_1\kappa_1\tau_1^2 - h_1\kappa_1'' - 3\tau_1'\Big)\overrightarrow{N}^* \\
&\quad +\Big(g_1\kappa_1'' - g\kappa_1^{2'} - g\kappa_1\kappa_1' - g\tau_1^{2'} - g\tau_1\tau_1' - g_1\kappa_1^3 - g_1\kappa_1\tau_1^2 - \kappa_1^2 - 3\tau_1^2\Big)\overrightarrow{B}^*
\end{aligned}
$$

eşitliği bulunur. Burada X_3, Y_3 ve Z_3 katsayıları

$$
\begin{aligned}
X_3 &= g_1\tau_1'' - g_1\kappa_1^2\tau_1 - g_1\tau_1^3 + h_1\tau_1\tau_1' + h_1\kappa_1^{2'} + h_1\tau_1^{2'} + h_1\kappa_1\kappa_1' + 2\kappa_1\tau_1 \\
Y_3 &= g\tau_1^3 + g\kappa_1^2\tau_1 - g\tau_1'' + h_1\kappa_1^3 + h_1\kappa_1\tau_1^2 - h_1\kappa_1'' - 3\tau_1' \\
Z_3 &= g_1\kappa_1'' - g\kappa_1^{2'} - g\kappa_1\kappa_1' - g\tau_1^{2'} - g\tau_1\tau_1' - g_1\kappa_1^3 - g_1\kappa_1\tau_1^2 - \kappa_1^2 - 3\tau_1^2
\end{aligned}
$$

şeklinde alınırsa β_2', β_2'' ve β_3''' ifadelerinin determinantı

$$
\begin{aligned}
\det(\beta_2', \beta_2'', \beta_2''') &= (g_1\tau_1' + h_1\kappa_1^2 + h_1\tau_1^2)(Y_3(1+g_1\kappa_1) + Z_3(g\tau_1 + h_1\kappa_1)) - (g_1\kappa_1' - g\kappa_1^2 - g\tau_1^2) \\
&\quad (Y_3 g_1\tau_1 - X_3(g\tau_1 + h_1\kappa_1)) + (g\tau_1' + 2\tau_1 + h_1\kappa_1')(X_3(1+g_1\kappa_1) - Z_3 g_1\tau_1)
\end{aligned} \tag{3.1.10}
$$

olur. (2.1.2) eşitliğinde (3.1.9) ve (3.1.10) eşitlikleri yerlerine yazılırsa eğrinin torsiyonu

$$\Big(g_1\tau_1' + h_1\kappa_1^2 + h_1\tau_1^2\Big)\Big(Y_3(1+g_1\kappa_1) + Z_3(g\tau_1 + h_1\kappa_1)\Big) - \Big(g_1\kappa_1' - g\kappa_1^2 - g\tau_1^2\Big).$$

$$\tau_{\beta_2} = \frac{\Big(Y_3 g_1\tau_1 - X_3(g\tau_1 + h_1\kappa_1)\Big) + \Big(g\tau_1' + 2\tau_1 + h_1\kappa_1'\Big)\Big(X_3(1+g_1\kappa_1) - Z_3 g_1\tau_1\Big)}{{X_2}^2 + {Y_2}^2 + {Z_2}^2}$$

elde edilir. □

Sonuç 3.1.3 β_2 eğrisinin $\overrightarrow{T}_{\beta_2}, \overrightarrow{N}_{\beta_2}, \overrightarrow{B}_{\beta_2}$ Frenet vektörlerinin α eğrisinin Frenet vektörleri cinsinden eşitleri

$$\overrightarrow{T}_{\beta_2} = \frac{\begin{array}{c}(\frac{\lambda\kappa\tau + h(\kappa^2+\tau^2)}{\lambda\kappa\sqrt{\kappa^2+\tau^2}} + \frac{g\kappa^2(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3})\overrightarrow{T} + (\frac{(h\kappa\tau - \kappa^2(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3})\overrightarrow{N} \\ +(\frac{\lambda\kappa^2 - (f+\lambda)(\kappa^2+\tau^2)}{\lambda\kappa\sqrt{\kappa^2+\tau^2}} - \frac{g\kappa\tau(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3})\overrightarrow{B}\end{array}}{\sqrt{(\frac{(h\kappa\tau - \kappa^2(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3})^2 + (\frac{h\kappa + \tau(f+\lambda)}{\lambda\kappa} + \frac{g\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)})^2 + (\frac{h\tau - f\kappa}{\lambda\kappa})^2}}$$

$$
\begin{aligned}
&\Big(X_2(\frac{\lambda\kappa^2-(\tau^2+\kappa^2)(f+\lambda)}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}-\frac{g\kappa\tau(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3})-\frac{Z_2\kappa^2(h\tau-\kappa(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda\sqrt{\kappa^2+\tau^2}}\\
&-\frac{Y_2\kappa(h\tau^2-\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)^2}\Big)\overrightarrow{T}\\
&+\Big(\frac{Y_2\lambda\kappa+Y_2(h\tau-\kappa(f+\lambda))+Z_2(h\kappa+\tau(f+\lambda))}{\lambda\kappa}+\frac{Z_2g\kappa(\frac{\tau}{\kappa})'}{\lambda\sqrt{\kappa^2+\tau^2}}\Big)\overrightarrow{N}\\
&+\Big(\frac{Z_2(h\tau^2-\kappa(f+\lambda))(\frac{\tau}{\kappa})'\kappa-Y_2(h\tau-\kappa(f+\lambda))(\frac{\tau}{\kappa})'\kappa^2}{\lambda(\kappa^2+\tau^2)^2}-X_2(\frac{\lambda\kappa\tau-h(\kappa^2+\tau^2)}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\\
\overrightarrow{N}_{\beta_2} \;=\;& \frac{-\frac{g\kappa^2(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3})\Big)\overrightarrow{B}}{\Big((\frac{(h\kappa\tau-\kappa^2(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3})^2+(\frac{(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))+g\kappa^2(\frac{\tau}{\kappa})'}{\lambda\kappa(\kappa^2+\tau^2)})^2 +(\frac{\lambda\kappa+h\tau-\kappa(f+\lambda)}{\lambda\kappa})^2\Big)^{\frac{1}{2}}\Big({X_2}^2+{Y_2}^2+{Z_2}^2\Big)^{\frac{1}{2}}}
\end{aligned}
$$

$$
\overrightarrow{B}_{\beta_2} = \frac{X_2+Y_2+Z_2}{\sqrt{{X_2}^2+{Y_2}^2+{Z_2}^2}}
$$

şeklinde verilir. Burada X_2, Y_2 ve Z_2

$$
\begin{aligned}
X_2 \;=\;& \Big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\frac{((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}\Big)\Big(g\frac{h\kappa+\tau(f+\lambda)}{\lambda\kappa}+\frac{g^2\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}\Big)+(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'(\frac{g(h\tau-f\kappa)}{\lambda\kappa})\\
&+\frac{(h\kappa\tau-\kappa^2(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}\Big(\frac{2\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}-g(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\Big)+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\\
&+\frac{2\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}
\end{aligned}
$$

$$Y_2 = \Big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2} + \frac{((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}\Big)\Big(\frac{g\kappa(h\tau-\kappa(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3} + \frac{(f+\lambda)\big(h(\tau^2-\kappa^2)-\kappa\tau(f+\lambda)\big)}{\kappa^2+\tau^2}\Big)$$

$$+\frac{(f\kappa-h\tau)(\kappa(f+\lambda)-h\tau)}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})' - \frac{(h\kappa\tau-\kappa^2(f+\lambda))^2(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)^2}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'$$

$$Z_2 = \Big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2} + (\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})^2\Big)\Big(\frac{g(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3} + \frac{(h\kappa+\tau(f+\lambda))^2}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\Big)$$

$$+\frac{(f+\lambda)(h(\tau^2-\kappa^2)-\kappa\tau(f+\lambda))+h^2\kappa}{\kappa^2+\tau^2}\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'$$

$$-(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big) - \frac{2\kappa^3(h\tau-\kappa(f+\lambda))((\frac{\tau}{\kappa})')^2}{\lambda(\sqrt{\kappa^2+\tau^2})^5}$$

şeklinde birer katsayıdır.

Sonuç 3.1.4 β_2 eğrisinin κ_{β_2} ve τ_{β_2} eğriliklerinin α eğrisinin eğrilikleri cinsinden yazılışı

$$\kappa_{\beta_2} = \frac{\sqrt{X_2+Y_2+Z_2}}{\Bigg(\big(\frac{(h\kappa\tau-\kappa^2(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}\big)^2+\big(\frac{h\kappa+\tau(f+\lambda)}{\lambda\kappa}+\frac{g\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}\big)^2+\big(\frac{h\tau-f\kappa}{\lambda\kappa}\big)^2\Bigg)^{\frac{3}{2}}}$$

$$\tau_{\beta_2} = \frac{\begin{aligned}&\Big(\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'+\frac{(h\kappa+\tau(f+\lambda))((\frac{\tau}{\kappa})')^2}{\lambda^4(\sqrt{\kappa^2+\tau^2})^3}\Big)\Big(Y_3\frac{h\tau-f\kappa}{\lambda\kappa}\\&+Z_3\big(\frac{g\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}+\frac{h\kappa+\tau(f+\lambda)}{\lambda\kappa}\big)\Big)-\Big(\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'-g\big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}\\&+\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2}\big)\Big)\Big(\frac{Y_3(h\kappa\tau-\kappa^2(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}-X_3\big(\frac{g\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}+\frac{h\kappa+\tau(f+\lambda)}{\lambda\kappa}\big)\Big)\\&+\Big(g\big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'+\frac{2\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\Big)\Big(X_3\frac{h\tau-f\kappa}{\lambda\kappa}\\&-\frac{Z_3\kappa(h\tau-\kappa(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}\Big)\end{aligned}}{X_2+Y_2+Z_2}$$

şeklinde verilir. Burada X_2, Y_2, Z_2, X_3, Y_3 ve Z_3

$$\begin{aligned}X_2 = {}&\Big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\frac{((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}\Big)\Big(\frac{gh\kappa+g\tau(f+\lambda)}{\lambda\kappa}+\frac{g^2\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}\Big)+\frac{g(h\tau-f\kappa)}{\lambda\kappa}\big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\\&+\frac{(h\kappa\tau-\kappa^2(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}\Big(\frac{2\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}-g\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\Big)+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\\&+\frac{2\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}\end{aligned}$$

$$
\begin{aligned}
Y_2 \;&=\; \Big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\frac{((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}\Big)\Big(\frac{g\kappa(h\tau-\kappa(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}+\frac{(f+\lambda)\big(h(\tau^2-\kappa^2)-\kappa\tau(f+\lambda)\big)}{\kappa^2+\tau^2}\Big) \\
&+\frac{(f\kappa-h\tau)(\kappa(f+\lambda)-h\tau)}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'-\frac{(h\kappa\tau-\kappa^2(f+\lambda))^2(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)^2}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'
\end{aligned}
$$

$$
\begin{aligned}
Z_2 \;&=\; \Big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})^2\Big)\Big(\frac{g(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}+\frac{(h\kappa+\tau(f+\lambda))^2}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}\Big) \\
&+\frac{(f+\lambda)(h(\tau^2-\kappa^2)-\kappa\tau(f+\lambda))+h^2\kappa}{\kappa^2+\tau^2}\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})' \\
&-(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big)-\frac{2\kappa^3(h\tau-\kappa(f+\lambda))((\frac{\tau}{\kappa})')^2}{\lambda(\sqrt{\kappa^2+\tau^2})^5}
\end{aligned}
$$

$$
\begin{aligned}
X_3 \;&=\; \frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big((\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2})'+(\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2})'+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})' \\
&+\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big)+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big((\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})''-\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}. \\
&(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2})\Big)+\frac{2(\frac{\tau}{\kappa})'}{\lambda^2\sqrt{\kappa^2+\tau^2}}
\end{aligned}
$$

$$
\begin{aligned}
Y_3 &= g\Big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2}\big)-\big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)''\Big)-\big(\frac{3\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}\big)' \\
&\quad +\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)^2\big)-\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''\Big)
\end{aligned}
$$

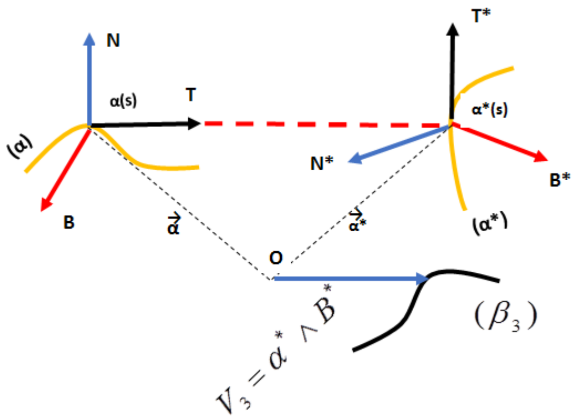
$$
\begin{aligned}
Z_3 &= -g\Big(\big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}\big)'+\big(\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2}\big)'+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'+\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\Big) \\
&\quad +\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big(\big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)''-\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}+\frac{((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}\big)\Big) \\
&\quad -\frac{\kappa^2+\tau^2}{\lambda^2\kappa^2}-\big(\frac{3\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)}\big)^2
\end{aligned}
$$

şeklinde birer katsayıdır.

Tanım 3.1.3 İnvolüt eğrisinin $\overrightarrow{B}^*$ binormal vektörünün $\overrightarrow{V_3} = \alpha^* \wedge \overrightarrow{B}^*$ vektörel moment vektörünün çizdiği eğri β_3 ile gösterilsin. Bu eğri

$$\beta_3 = g_1 \overrightarrow{T}^* - g\overrightarrow{N}^*$$

şeklindedir (Şekil 3.3).



Şekil 3.3: V_3 Vektörel Moment Vektörünün Çizdiği β_3 Eğrisi

Teorem 3.1.5 β_3 eğrisinin Frenet vektörleri $\overrightarrow{T}_{\beta_3}, \overrightarrow{N}_{\beta_3}, \overrightarrow{B}_{\beta_3}$ ile gösterilsin. Bu vektörlerin involüt eğrisinin Frenet vektörleri cinsinden karşılıkları

$$\overrightarrow{T}_{\beta_3} = \frac{h_1\tau_1\overrightarrow{T}^* - \overrightarrow{N}^* - g\tau_1\overrightarrow{B}^*}{\sqrt{1+\tau_1^2(g^2+h_1^2)}}$$

$$\overrightarrow{N}_{\beta_3} = \frac{\begin{array}{l}\Big(g^2\tau_1^2(\kappa_1 - g_1\tau_1^2) - gh_1\tau_1^3(g_1\kappa_1+1) + \tau_1^2(h_1^2\kappa_1 - g_1) + h_1\tau_1' + \kappa_1\Big)\overrightarrow{T}^* + \Big(g\tau_1^2(h_1^2\tau_1^2 \\ +g_1\kappa_1 + 2) + g^2\tau_1^3(g\tau_1 + h_1\kappa_1) + \tau_1\tau_1'(g^2+h_1^2) + h_1\tau_1^3(h_1^2\kappa_1 - g_1) + h_1\kappa_1\tau_1\Big)\overrightarrow{N}^* \\ -\Big(g\tau_1' + g\tau_1^3(g+g_1h_1\tau_1) + (g_1\kappa_1\tau_1 + 2\tau_1)(h_1^2\tau_1^2+1)\Big)\overrightarrow{B}^*\end{array}}{\begin{array}{l}\Bigg(\bigg(\Big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\Big)^2 + \Big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\Big)^2 + \Big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\Big)^2\bigg)\Big(\tau_1^2(g^2+h_1^2)+1\Big)\Bigg)^{\frac{1}{2}}\end{array}}$$

$$\overrightarrow{B}_{\beta_3} = \frac{\begin{array}{l}\Big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\Big)\overrightarrow{T}^* + \Big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 + 2h_1\tau_1^2\Big)\overrightarrow{N}^* \\ + \Big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\Big)\overrightarrow{B}^*\end{array}}{\left(\begin{array}{l}\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\end{array}\right)^{\frac{1}{2}}}$$

bağıntılarıyla verilir.

İspat. β_3 eğrisinin türevi alınırsa

$$\begin{aligned}
\beta_3' &= g_1'\overrightarrow{T}^* + g_1\overrightarrow{T}^{*\prime} - g'\overrightarrow{N}^* - g\overrightarrow{N}^{*\prime} \\
&= g_1'\overrightarrow{T}^* + g_1(\kappa_1\overrightarrow{N}^*) - g'\overrightarrow{N}^* - g(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) \\
&= (g_1' + g\kappa_1)\overrightarrow{T}^* + (g_1\kappa_1 - g')\overrightarrow{N}^* - g\tau_1\overrightarrow{B}^* \\
&= h_1\tau_1\overrightarrow{T}^* - \overrightarrow{N}^* - g\tau_1\overrightarrow{B}^* \qquad (3.1.11)
\end{aligned}$$

bulunur. Buradan norm alınırsa

$$\|\beta_3'\| = \sqrt{\tau_1^2(g^2 + h_1^2) + 1} \qquad (3.1.12)$$

olur. (2.1.1) eşitliğinde (3.1.11) ve (3.1.12) eşitlikleri yerlerine yazıldığında $\overrightarrow{T}_{\beta_3}$ teğet vektörü

$$\overrightarrow{T}_{\beta_3} = \frac{h_1\tau_1\overrightarrow{T}^* - \overrightarrow{N}^* - g\tau_1\overrightarrow{B}^*}{\sqrt{\tau_1^2(g^2 + h_1^2) + 1}}$$

şeklinde olur. β_3' ifadesinin türevi alınırsa

$$\begin{aligned}
\beta_3'' &= (h_1'\tau_1 + h_1\tau_1')\overrightarrow{T}^* + h_1\tau_1\overrightarrow{T}^{*\prime} - \overrightarrow{N}^{*\prime} - (g'\tau_1 + g\tau_1')\overrightarrow{B}^* - g\tau_1\overrightarrow{B}^{*\prime} \\
&= (h_1'\tau_1 + h_1\tau_1')\overrightarrow{T}^* + h_1\tau_1(\kappa_1\overrightarrow{N}^*) - (-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) - (g'\tau_1 + g\tau_1')\overrightarrow{B}^* - g\tau_1(-\tau_1\overrightarrow{N}^*) \\
&= (h_1'\tau_1 + h_1\tau_1' + \kappa_1)\overrightarrow{T}^* + (h_1\kappa_1\tau_1 + g\tau_1^2)\overrightarrow{N}^* - (g'\tau_1 + g\tau_1' + \tau_1)\overrightarrow{B}^* \\
&= (h_1\tau_1' - g_1\tau_1^2 + \kappa_1)\overrightarrow{T}^* + (g\tau_1^2 + h_1\kappa_1\tau_1)\overrightarrow{N}^* - (g\tau_1' + g_1\kappa_1\tau_1 + 2\tau_1)\overrightarrow{B}^*
\end{aligned}$$

olur. β_3' ve β_3'' eşitliklerinin vektörel çarpıldığında

$$\begin{aligned}
\beta_3' \wedge \beta_3'' &= (g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1)\overrightarrow{T}^* + (gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\
&\quad +2h_1\tau_1^2)\overrightarrow{N}^* + (gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1)\overrightarrow{B}^* \qquad (3.1.13)
\end{aligned}$$

bulunur. Bu vektörel çarpımın normu

$$\begin{aligned}
\|\beta_3' \wedge \beta_3''\| &= \Big(\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\
&\quad +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\Big)^{\frac{1}{2}} \qquad (3.1.14)
\end{aligned}$$

şeklinde olur. (2.1.1) eşitliğinde (3.1.13) ve (3.1.14) yerlerine yazıldığında eğrinin binormal vektörü

$$\overrightarrow{B}_{\beta_3} = \frac{\begin{array}{l}\Big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\Big)\overrightarrow{T}^* + \Big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 + 2h_1\tau_1^2\Big)\overrightarrow{N}^* \\ +\Big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\Big)\overrightarrow{B}^*\end{array}}{\left(\begin{array}{l}\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\end{array}\right)^{\frac{1}{2}}}$$

bulunur. (2.1.1) eşitliğinde $\overrightarrow{T}_{\beta_3}$ ve $\overrightarrow{B}_{\beta_3}$ vektöleri yerlerine yazıldığında eğrinin asli normal vektörü

$$\overrightarrow{N}_{\beta_3} = \frac{\begin{array}{l}\Big(g^2\tau_1^2(\kappa_1 - g_1\tau_1^2) - gh_1\tau_1^3(g_1\kappa_1+1) + \tau_1^2(h_1^2\kappa_1 - g_1) + h_1\tau_1' + \kappa_1\Big)\overrightarrow{T}^* + \Big(g\tau_1^2(h_1^2\tau_1^2 \\ +g_1\kappa_1 + 2) + g^2\tau_1^3(g\tau_1 + h_1\kappa_1) + \tau_1\tau_1'(g^2+h_1^2) + h_1\tau_1^3(h_1^2\kappa_1 - g_1) + h_1\kappa_1\tau_1\Big)\overrightarrow{N}^* \\ -\Big(g\tau_1' + g\tau_1^3(g+g_1h_1\tau_1) + (g_1\kappa_1\tau_1 + 2\tau_1)(h_1^2\tau_1^2+1)\Big)\overrightarrow{B}^*\end{array}}{\begin{array}{l}\Bigg(\Big(\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\Big)\Big(\tau_1^2(g^2+h_1^2)+1\Big)\Bigg)^{\frac{1}{2}}\end{array}}$$

elde edilir. □

Teorem 3.1.6 β_3 eğrisinin eğriliği ve torsiyonu κ_{β_3} ve τ_{β_3} ile gösterilsin. Bu eğriliklerin involüt eğrisinin eğrilikleri cinsinden karşılıkları

$$\kappa_{\beta_3} = \frac{\begin{array}{l}\Big(\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\Big)^{\frac{1}{2}}\end{array}}{\Big(\tau_1^2(g^2+h_1^2)+1\Big)^{\frac{3}{2}}}$$

$$\tau_{\beta_3} = \frac{\begin{array}{l}\big(g\tau_1^2 + h_1\kappa_1\tau_1\big)\big(h_1\tau_1F_3 + g\tau_1F_1\big) + \big(h_1\tau_1' - g\tau_1^2 + \kappa_1\big)\big(F_3 - g\tau_1F_2\big) \\ +\big(g\tau_1' + g_1\kappa_1\tau_1 + 2\tau_1\big)\big(F_1 + h_1\tau_1F_2\big)\end{array}}{\begin{array}{l}\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\end{array}}$$

şeklinde verilir. Burada F_1, F_2 ve F_3

$$\begin{cases} F_1 = & \kappa_1' - g_1\tau_1\tau_1' - g_1\tau_1^{2'} - h_1\tau_1^3 - h_1\kappa_1^2\tau_1 + h_1\tau_1'' \\ F_2 = & g\tau_1^{2'} + g\tau_1\tau_1' + h_1\kappa_1'\tau_1 + \kappa_1^2 + 3\tau_1^3 + 2h_1\kappa_1\tau_1' \\ F_3 = & g\tau_1^3 + g\kappa_1^2\tau_1 - g\tau_1'' - g_1\kappa_1'\tau_1 - 2g_1\kappa_1\tau_1' - 3\tau_1' \end{cases}$$

şeklinde birer katsayıdır.

İspat. (2.1.2) ifadesinde (3.1.12) ve (3.1.14) bağıntıları yerlerine yazıldığında eğrinin eğriliği

$$\kappa_{\beta_3} = \frac{\begin{array}{c}\Big(\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\Big)^{\frac{1}{2}}\end{array}}{\Big(\tau_1^2(g^2 + h_1^2) + 1\Big)^{\frac{3}{2}}}$$

şeklindedir. β_3'' ifadesinin türevi alınırsa

$$\begin{aligned} \beta_3''' = {} & (h_1'\tau_1' + h_1\tau_1'' - g_1'\tau_1^2 - g_1\tau_1^{2'} + \kappa_1')\overrightarrow{T}^* + (h_1\tau_1' - g_1\tau_1^2 + \kappa_1)\overrightarrow{T}^{*'} + (g'\tau_1^2 + g\tau_1^{2'} \\ & +h_1'\kappa_1\tau_+ h_1\kappa_1'\tau_1 + h_1\kappa_1\tau_1')\overrightarrow{N}^* + (g\tau_1^2 + h_1\kappa_1\tau_1)\overrightarrow{N}^{*'} - (g'\tau_1' + g\tau_1'' \\ & +g_1'\kappa_1\tau_1 + g_1\kappa_1'\tau_1 + g_1\kappa_1\tau_1' + 2\tau_1')\overrightarrow{B}^* - (g\tau_1' + g_1\kappa_1\tau_1 + 2\tau_1)\overrightarrow{B}^{*'} \end{aligned}$$

$$
\begin{aligned}
&= (h_1'\tau_1' + h_1\tau_1'' - g_1'\tau_1^2 - g_1\tau_1^{2'} + \kappa_1')\overrightarrow{T}^* + (h_1\tau_1' - g_1\tau_1^2 + \kappa_1)(\kappa_1\overrightarrow{N}^*) + (g'\tau_1^2 + g\tau_1^{2'} \\
&\quad + h_1'\kappa_1\tau_1 + h_1\kappa_1'\tau_1 + h_1\kappa_1\tau_1')\overrightarrow{N}^* + (g\tau_1^2 + h_1\kappa_1\tau_1)(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) - (g'\tau_1' + g\tau_1'' \\
&\quad + g_1'\kappa_1\tau_1 + g_1\kappa_1'\tau_1 + g_1\kappa_1\tau_1' + 2\tau_1')\overrightarrow{B}^* - (g\tau_1' + g_1\kappa_1\tau_1 + 2\tau_1)(-\tau_1\overrightarrow{N}^*) \\
&= (h_1'\tau_1' + h_1\tau_1'' - g_1'\tau_1^2 - g_1\tau_1^{2'} + \kappa_1' - g\kappa_1\tau_1^2 - h_1\kappa_1^2\tau_1)\overrightarrow{T}^* + (h_1\kappa_1\tau_1' - g_1\kappa_1\tau_1^2 + \kappa_1^2 \\
&\quad + g'\tau_1^2 + g\tau_1^{2'} + h_1'\kappa_1\tau_1 + h_1\kappa_1'\tau_1 + h_1\kappa_1\tau_1' + g\tau_1\tau_1' + g_1\kappa_1\tau_1^2 + 2\tau_1^2)\overrightarrow{N}^* + (g\tau_1^3 \\
&\quad + h_1\kappa_1\tau_1^2 - g'\tau_1' - g\tau_1'' - g_1'\kappa_1\tau_1 - g_1\kappa_1'\tau_1 - g_1\kappa_1\tau_1' - 2\tau_1')\overrightarrow{B}^* \\
&= ((-g_1\tau_1)\tau_1' + h_1\tau_1'' - (h_1\tau_1 - g\kappa_1)\tau_1^2 - g_1\tau_1^{2'} + \kappa_1' - g\kappa_1\tau_1^2 - h_1\kappa_1^2\tau_1)\overrightarrow{T}^* \\
&\quad + (h_1\kappa_1\tau_1' - g_1\kappa_1\tau_1^2 + \kappa_1^2 + (1 + g_1\kappa_1)\tau_1^2 + g\tau_1^{2'} - (g_1\tau_1)\kappa_1\tau_1 + h_1\kappa_1'\tau_1 + h_1\kappa_1\tau_1' \\
&\quad + g\tau_1\tau_1' + g_1\kappa_1\tau_1^2 + 2\tau_1^2)\overrightarrow{N}^* + (g\tau_1^3 + h_1\kappa_1\tau_1^2 - (1 + g_1\kappa_1)\tau_1' - g\tau_1'' - (h_1\tau_1 - g\kappa_1)\kappa_1\tau_1 \\
&\quad - g_1\kappa_1'\tau_1 - g_1\kappa_1\tau_1' - 2\tau_1')\overrightarrow{B}^* \\
&= \Big(\kappa_1' - g_1\tau_1\tau_1' - g_1\tau_1^{2'} - h_1\tau_1^3 - h_1\kappa_1^2\tau_1 + h_1\tau_1''\Big)\overrightarrow{T}^* + \Big(g\tau_1^{2'} + g\tau_1\tau_1' + h_1\kappa_1'\tau_1 + \kappa_1^2 \\
&\quad + 3\tau_1^3 + 2h_1\kappa_1\tau_1'\Big)\overrightarrow{N}^* + \Big(g\tau_1^3 + g\kappa_1^2\tau_1 - g\tau_1'' - g_1\kappa_1'\tau_1 - 2g_1\kappa_1\tau_1' - 3\tau_1'\Big)\overrightarrow{B}^*
\end{aligned}
$$

olur. Burada F_1, F_2 ve F_3

$$
\begin{cases}
F_1 = & \kappa_1' - g_1\tau_1\tau_1' - g_1\tau_1^{2'} - h_1\tau_1^3 - h_1\kappa_1^2\tau_1 + h_1\tau_1'' \\
F_2 = & g\tau_1^{2'} + g\tau_1\tau_1' + h_1\kappa_1'\tau_1 + \kappa_1^2 + 3\tau_1^3 + 2h_1\kappa_1\tau_1' \\
F_3 = & g\tau_1^3 + g\kappa_1^2\tau_1 - g\tau_1'' - g_1\kappa_1'\tau_1 - 2g_1\kappa_1\tau_1' - 3\tau_1'
\end{cases}
$$

olarak alınırsa β_3', β_3'' ve β_3''' ifadelerinin determinantı

$$\begin{aligned}\det(\beta_3', \beta_3'', \beta_3''') &= \big(g\tau_1^2 + h_1\kappa_1\tau_1\big)\big(h_1\tau_1F_3 + g\tau_1F_1\big) + \big(h_1\tau_1' - g\tau_1^2 + \kappa_1\big)\big(F_3 - g\tau_1F_2\big) \\ &\quad + \big(g\tau_1' + g_1\kappa_1\tau_1 + 2\tau_1\big)\big(F_1 + h_1\tau_1F_2\big)\end{aligned}$$

bulunur. (2.1.2) ifadesinde $\det(\beta_3', \beta_3'', \beta_3''')$ ve (3.1.14) bağıntıları yerlerine yazılırsa eğrinin torsiyonu

$$\tau_{\beta_3} = \frac{\begin{array}{l}\big(g\tau_1^2 + h_1\kappa_1\tau_1\big)\big(h_1\tau_1F_3 + g\tau_1F_1\big) + \big(h_1\tau_1' - g\tau_1^2 + \kappa_1\big)\big(F_3 - g\tau_1F_2\big) \\ +\big(g\tau_1' + g_1\kappa_1\tau_1 + 2\tau_1\big)\big(F_1 + h_1\tau_1F_2\big)\end{array}}{\begin{array}{l}\big(g\tau_1' + g^2\tau_1^3 + gh_1\kappa_1\tau_1^2 + g_1\kappa_1\tau_1 + 2\tau_1\big)^2 + \big(gg_1\tau_1^3 - g\kappa_1\tau_1 + g_1h_1\kappa_1\tau_1^2 \\ +2h_1\tau_1^2\big)^2 + \big(gh_1\tau_1^3 - g_1\tau_1^2 + h_1^2\kappa_1\tau_1^2 + h_1\tau_1' + \kappa_1\big)^2\end{array}}$$

elde edilir. $\square$

Sonuç 3.1.5 β_3 eğrisinin $\overrightarrow{T}_{\beta_3}, \overrightarrow{N}_{\beta_3}, \overrightarrow{B}_{\beta_3}$ Frenet vektörlerinin α eğrisinin Frenet vektörleri cinsinden eşitleri

$$\overrightarrow{T}_{\beta_3} = \frac{-\big(\lambda(\kappa^3+\kappa\tau^2)+g\kappa\tau(\frac{\tau}{\kappa})'\big)\overrightarrow{T}+(h\kappa\tau-\kappa^2(f+\lambda))(\frac{\tau}{\kappa})'\overrightarrow{N}-\big(\lambda(\kappa^2\tau+\tau^3)+g\kappa^2(\frac{\tau}{\kappa})'\big)\overrightarrow{B}}{\sqrt{\kappa^2+\tau^2}((\frac{\tau}{\kappa})')\kappa\sqrt{g^2(\kappa^2+\tau^2)+(h\kappa+\tau(f+\lambda))^2+\lambda^2(\kappa^2+\tau^2)^2}}$$

$$\overrightarrow{N}_{\beta_3} = \frac{\frac{1}{\sqrt{\kappa^2+\tau^2}}\Big((Z_4\tau-Y_4\kappa)\overrightarrow{T}+\sqrt{\kappa^2+\tau^2}X_4\overrightarrow{N}+(Y_4\tau+Z_4\kappa)\overrightarrow{B}\Big)}{\begin{array}{l}\Bigg(\Big(\frac{g((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa^2+\kappa\tau(f+\lambda)\big)}{\lambda^3(\kappa^2+\tau^2)^3}+\frac{(\frac{\tau}{\kappa})'(h\tau-\kappa(f-\lambda))}{\lambda^2(\kappa^2+\tau^2)} \\ +g(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big)^2+\Big(\frac{(h\tau-\kappa(f+\lambda))((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa\tau-\kappa^2(f+\lambda)\big)}{\lambda^3(\sqrt{\kappa^2+\tau^2})^7} \\ +\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}(\frac{2(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{(\sqrt{\kappa^2+\tau^2})^3}-\frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\Big)^2+\Big((-\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}} \\ +\frac{(h\kappa+\tau(f+\lambda))\big(g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa+\tau(f+\lambda)\big)}{\lambda\kappa(\sqrt{\kappa^2+\tau^2})^3}\big)\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2} \\ +\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)^2\Bigg)^{\frac{1}{2}}. \\ \Bigg(\frac{((\frac{\tau}{\kappa})')^2\kappa^2\big(g^2(\kappa^2+\tau^2)+(h\kappa+\tau(f+\lambda))^2\big)}{\lambda^2(\kappa^2+\tau^2)^2}+1\Bigg)^{\frac{1}{2}}\end{array}}$$

$$
\overrightarrow{B}_{\beta_3} = \frac{\begin{aligned}&\Big(\frac{((\frac{\tau}{\kappa})')^2\Big((f+\lambda)\big(g\kappa^3(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa^2+\kappa\tau(f+\lambda))\big)-\lambda(h\kappa^2\tau^2-\kappa^3\tau(f+\lambda))\Big)}{\lambda^3(\sqrt{\kappa^2+\tau^2})^5}\\&-\frac{(\frac{\tau}{\kappa})'\kappa^2}{\lambda(\kappa^2+\tau^2)}(\frac{2\lambda(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}-\frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})+\frac{\tau\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\\&+\frac{h\kappa\tau+\tau^2(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big)\overrightarrow{T}\\&+\Big(g(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+\frac{g^2\kappa^3((\frac{\tau}{\kappa})')^3(h\kappa+\tau(f\lambda))}{\lambda^3(\kappa^2+\tau^2)^3}+\frac{(\frac{\tau}{\kappa})'(h\tau-\kappa(f-\lambda))}{\lambda^2(\kappa^2+\tau^2)}\Big)\overrightarrow{N}\\&+\Big(\frac{h\kappa((\frac{\tau}{\kappa})')^2\big(g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)(h\kappa+\tau(f+\lambda))\big)-(h\kappa\tau-\kappa^2(f+\lambda))((\frac{\tau}{\kappa})')^2\lambda\kappa^2}{\lambda^3(\sqrt{\kappa^2+\tau^2})^5}\\&+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda}+\frac{(\frac{\tau}{\kappa})'\kappa\tau}{\lambda(\kappa^2+\tau^2)}(\frac{2(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}-\frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\\&+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big)\overrightarrow{B}\end{aligned}}{\begin{aligned}&\Bigg(\Big(\frac{g((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa^2+\kappa\tau(f+\lambda)\big)}{\lambda^3(\kappa^2+\tau^2)^3}+\frac{(\frac{\tau}{\kappa})'(h\tau-\kappa(f-\lambda))}{\lambda^2(\kappa^2+\tau^2)}\\&+g(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big)^2+\Big(\frac{(h\tau-\kappa(f+\lambda))((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa\tau-\kappa^2(f+\lambda)\big)}{\lambda^3(\sqrt{\kappa^2+\tau^2})^7}\\&+\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}(\frac{2(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{(\sqrt{\kappa^2+\tau^2})^3}-\frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\Big)^2+\Big((-\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\\&+\frac{(h\kappa+\tau(f+\lambda))\big(g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa+\tau(f+\lambda)\big)}{\lambda\kappa(\sqrt{\kappa^2+\tau^2})^3})\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2}\\&+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)^2\Bigg)^{\frac{1}{2}}\end{aligned}}
$$

bağıntılarıyla verilir. Burada X_4, Y_4 ve Z_4

$$
\begin{aligned}
X_4 &= \frac{g^2((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}-\frac{(h\tau-\kappa(f+\lambda))((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\sqrt{\kappa^2+\tau^2})^5}\Big)-\frac{((\frac{\tau}{\kappa})'\kappa)^3}{\lambda^3(\kappa^2+\tau^2)^3}. \\
&\quad \frac{\big(gh\kappa+g\tau(f+\lambda)\big)\big(h\tau-f\kappa\big)}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}+\frac{((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}\Big(\frac{(h\kappa+\tau(f+\lambda))^2}{\lambda\kappa\sqrt{\kappa^2+\tau^2}}-\frac{h\tau-\kappa(f+\lambda)}{\lambda\kappa}\Big) \\
&\quad +\frac{h\kappa+\tau(f+\lambda)}{\kappa^2+\tau^2}\Big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big)'+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}
\end{aligned}
$$

$$
\begin{aligned}
Y_4 &= \frac{g\kappa^2((\frac{\tau}{\kappa})')^2}{\lambda^2(\kappa^2+\tau^2)^2}\Big(\frac{(h\kappa+\tau(f+\lambda))^2((\frac{\tau}{\kappa})'\kappa)^2}{(\kappa^2+\tau^2)^3}+\frac{h\tau+\kappa(\lambda-f)}{\lambda\kappa}\Big) \\
&\quad +\frac{g^2\kappa^3((\frac{\tau}{\kappa})')^3}{\lambda^4(\kappa^2+\tau^2)^3}\Big(\frac{g\kappa(\frac{\tau}{\kappa})'}{\kappa^2+\tau^2}+\frac{h\kappa+\tau(f+\lambda)}{\kappa}\Big)+\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big)'. \\
&\quad \frac{g^2(\kappa^2+\tau^2)+(h\kappa+\tau(f+\lambda))}{\kappa^2+\tau^2}+\frac{(h\kappa+\tau(f+\lambda))((\frac{\tau}{\kappa})'\kappa)^3}{\lambda^3(\sqrt{\kappa^2+\tau^2})^7}\Big(\frac{(h\kappa+\tau(f+\lambda))^2}{\lambda\kappa(\sqrt{\kappa^2+\tau^2})^3} \\
&\quad -\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big)+\frac{(h\kappa+\tau(f+\lambda))((\frac{\tau}{\kappa})')}{\lambda^2(\kappa^2+\tau^2)}
\end{aligned}
$$

$$
\begin{aligned}
Z_4 &= -\frac{g\kappa^2((\frac{\tau}{\kappa})')^3}{\lambda^4(\kappa^2+\tau^2)^5}\Big(g\lambda\kappa(\kappa^2+\tau^2)^2+\big(h(f+\lambda)(\tau^2-\kappa^2)+\kappa\tau(h^2-(f+\lambda))\big)\big((\frac{\tau}{\kappa})'\kappa^2\big)\Big) \\
&\quad +\frac{2\lambda(h\kappa\tau-\kappa^2(f+\lambda))((\frac{\tau}{\kappa})')^2}{\lambda^4(\kappa^2+\tau^2)^4}\Big((h\kappa+\tau(f+\lambda))^2((\frac{\tau}{\kappa})'\kappa)^2+\lambda^2(\kappa^2+\tau^2)^3\Big) \\
&\quad +g\Big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big)'
\end{aligned}
$$

şeklinde birer katsayıdır.

Sonuç 3.1.6 β_3 eğrisinin κ_{β_3} ve τ_{β_3} eğriliklerinin α esas eğrinin eğrilikleri cinsinden yazılışı

$$\kappa_{\beta_3} = \frac{\begin{aligned}&\Bigg(\bigg(\frac{g((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})' + (\kappa^2+\tau^2)h\kappa^2 + \kappa\tau(f+\lambda)\big)}{\lambda^3(\kappa^2+\tau^2)^3} + \frac{(\frac{\tau}{\kappa})'(h\tau - \kappa(f-\lambda))}{\lambda^2(\kappa^2+\tau^2)} \\ &+g(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\bigg)^2 + \bigg(\frac{(h\tau-\kappa(f+\lambda))((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})' + (\kappa^2+\tau^2)h\kappa\tau - \kappa^2(f+\lambda)\big)}{\lambda^3(\sqrt{\kappa^2+\tau^2})^7} \\ &+\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}(\frac{2(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{(\sqrt{\kappa^2+\tau^2})^3} - \frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\bigg)^2 + \bigg(\big(-\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}} \\ &+\frac{(h\kappa+\tau(f+\lambda))\big(g\kappa^2(\frac{\tau}{\kappa})' + (\kappa^2+\tau^2)h\kappa + \tau(f+\lambda)\big)}{\lambda\kappa(\sqrt{\kappa^2+\tau^2})^3}\big)\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2} \\ &+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})' + \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\bigg)^2\Bigg)^{\frac{1}{2}}\end{aligned}}{\Bigg(\frac{((\frac{\tau}{\kappa})'\kappa)^2\Big(g^2(\kappa^2+\tau^2) + (h\kappa+\tau(f+\lambda))^2\Big)}{\lambda^2(\kappa^2+\tau^2)^2} + 1\Bigg)^{\frac{3}{2}}}$$

$$\Big(\frac{g((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}+\frac{(h\kappa+\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda^2(\kappa^2+\tau^2)}\Big)\Big(\frac{F_3(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}+\frac{g\kappa(\frac{\tau}{\kappa})'F_1}{\lambda(\kappa^2+\tau^2)}\Big)$$

$$+\Big(\frac{(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda(\sqrt{\kappa^2+\tau^2})^3}-\frac{g((\frac{\tau}{\kappa})'\kappa)^2}{\lambda^2(\kappa^2+\tau^2)^2}+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)\Big(F_3-\frac{g\kappa(\frac{\tau}{\kappa})'F_2}{\lambda(\kappa^2+\tau^2)}\Big)$$

$$+\Big(g(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+\frac{(h\tau-\kappa(f+\lambda))(\frac{\tau}{\kappa})'}{\lambda^2(\kappa^2+\tau^2)}+\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big).$$

$$\tau_{\beta_3} = \frac{\Big(F_1+\frac{(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'F_2}{\lambda(\sqrt{\kappa^2+\tau^2})^3}\Big)}{\begin{aligned}&\Big(\frac{g((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa^2+\kappa\tau(f+\lambda)\big)}{\lambda^3(\kappa^2+\tau^2)^3}+\frac{(\frac{\tau}{\kappa})'(h\tau-\kappa(f-\lambda))}{\lambda^2(\kappa^2+\tau^2)}\\&+g(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big)^2+\Big(\frac{(h\tau-\kappa(f+\lambda))((\frac{\tau}{\kappa})')^2\big(g\kappa^3(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa\tau-\kappa^2(f+\lambda)\big)}{\lambda^3(\sqrt{\kappa^2+\tau^2})^7}\\&+\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}(\frac{2(h\kappa^2+\kappa\tau(f+\lambda))(\frac{\tau}{\kappa})'}{(\sqrt{\kappa^2+\tau^2})^3}-\frac{g\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\Big)^2+\Big((-\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\\&+\frac{(h\kappa+\tau(f+\lambda))\big(g\kappa^2(\frac{\tau}{\kappa})'+(\kappa^2+\tau^2)h\kappa+\tau(f+\lambda)\big)}{\lambda\kappa(\sqrt{\kappa^2+\tau^2})^3})\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2}\\&+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)^2\end{aligned}}$$

bağıntılarıyla verilir. Burada F_1, F_2 ve F_3

$$F_1 = (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'-\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+(\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2})'\Big)$$

$$-\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big(\frac{((\frac{\tau}{\kappa})'\kappa)^3}{\lambda^3(\kappa^2+\tau^2)^3}+\frac{(\frac{\tau}{\kappa})'}{\lambda^3\kappa}+(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})''\Big)$$

$$F_2 = g\Big((\frac{((\frac{\tau}{\kappa})')^2\kappa^2}{\lambda^2(\kappa^2+\tau^2)^2})' + (\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big) + \frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big((\frac{\kappa^2+\tau^2}{\lambda\kappa})'.$$

$$(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}) + \frac{\kappa^2+\tau^2}{\lambda\kappa}(\frac{2(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big) + \frac{\kappa^2+\tau^2}{\lambda^2\kappa^2} + \frac{3((\frac{\tau}{\kappa})'\kappa)^3}{\lambda^3(\kappa^2+\tau^2)^3}$$

$$F_3 = g\Big(\frac{((\frac{\tau}{\kappa})'\kappa)^3}{\lambda^3(\kappa^2+\tau^2)^3} + \frac{(\frac{\tau}{\kappa})'}{\lambda^3\kappa} - (\frac{(\frac{\tau}{\kappa})'\kappa^2}{\lambda\kappa(\kappa^2+\tau^2)})''\Big) - \frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\Big((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'.$$

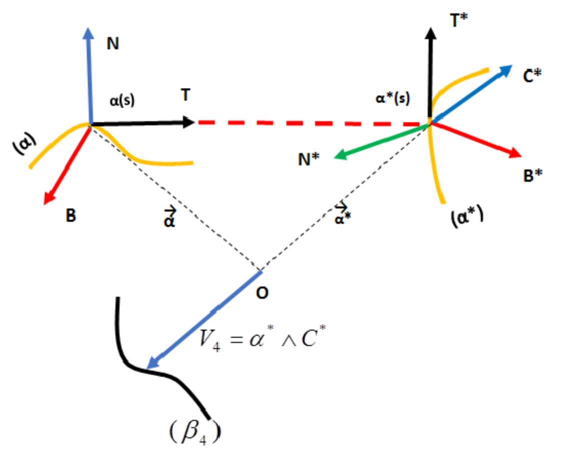
$$\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)} + \frac{2\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big) - 3(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'$$

şeklinde birer katsayıdır.

Tanım 3.1.4 İnvolüt eğrisinin $\overrightarrow{C}^*$ birim Darboux vektörünün $\overrightarrow{V_4} = \alpha^* \wedge \overrightarrow{C}^*$ vektörel moment vektörünün çizdiği eğri β_4 ile gösterilsin. Bu eğri

$$\beta_4 = \frac{g_1\kappa_1}{\sqrt{\kappa_1^2+\tau_1^2}}\overrightarrow{T}^* + \frac{h_1\tau_1 - g\kappa_1}{\sqrt{\kappa_1^2+\tau_1^2}}\overrightarrow{N}^* - \frac{g_1\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}\overrightarrow{B}^*$$

şeklindedir (Şekil 3.4).



Şekil 3.4: V_4 Vektörel Moment Vektörünün Çiziği B_4 Eğrisi

Teorem 3.1.7 β_4 eğrisinin Frenet vektörleri $\overrightarrow{T}_{\beta_4}, \overrightarrow{N}_{\beta_4}, \overrightarrow{B}_{\beta_4}$ ile gösterilsin. Bu vektörlerin involüt eğrisinin Frenet vektörleri cinsinden karşılıkları

$$\overrightarrow{T}_{\beta_4} = \frac{g_1(r^*\kappa_1)'\overrightarrow{T}^* + \big(h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1\big)\overrightarrow{N}^* - g_1(r^*\tau_1)'\overrightarrow{B}^*}{\big((g_1(r^*\kappa_1)')^2 + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)^2 + (g_1(r^*\tau_1)')^2\big)^{\frac{1}{2}}}$$

$$\overrightarrow{N}_{\beta_4} = \frac{\begin{array}{l}\Big(g_1(r^*\tau_1)'(g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)')) - (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)((r^*\kappa_1)' \\ (gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1))\Big)\overrightarrow{T}^* + \Big(g_1(r^*\kappa_1)'((r^*\kappa_1)'(gH_1 + g_1H_2) \\ -H_1(h_1(r^*\tau_1)' - r^*\kappa_1)) + g_1(r^*\tau_1)'((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1))\Big)\overrightarrow{N}^* \\ +\Big((h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1)) \\ +g_1(r^*\kappa_1)'(g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)'))\Big)\overrightarrow{B}^*\end{array}}{\begin{array}{l}\Big(((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1))^2 + (g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)'))^2 \\ +((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1))^2\Big)^{\frac{1}{2}}. \\ \Big((g_1(r^*\kappa_1')')^2 + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)^2 + (g_1(r^*\tau_1)')^2\Big)^{\frac{1}{2}}\end{array}}$$

$$\overrightarrow{B}_{\beta_4} = \frac{\begin{array}{l}\Big((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1)\Big)\overrightarrow{T}^* - \Big(g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)')\Big)\overrightarrow{N}^* \\ +\Big((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1)\Big)\overrightarrow{B}^*\end{array}}{\begin{array}{l}\big(((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1))^2 + (g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)'))^2 \\ +((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1))^2\big)^{\frac{1}{2}}\end{array}}$$

şeklinde verilir. Burada r^*, H_1, H_2, H_3

$$
\begin{aligned}
r^* &= \frac{1}{\sqrt{\kappa_1^2+\tau_1^2}} \\
H_1 &= g_1(r^*\kappa_1)'' + r^*\kappa_1^2 + h_1 r^*(\kappa_1'\tau_1 - \kappa_1\tau_1') \\
H_2 &= -(g(r^*\kappa_1)'' + 2(r^*\kappa_1)' - h_1(r^*\tau_1)'') \\
H_3 &= gr^*(\kappa_1\tau_1' - \kappa_1'\tau_1) - r^*\kappa_1\tau_1 - g_1(r^*\tau_1)''
\end{aligned}
$$

biçiminde birer katsayıdır.

İspat. β_4 eğrisininde $r^* = \dfrac{1}{\sqrt{\kappa_1^2+\tau_1^2}}$ olmak üzere türev alınırsa

$$
\begin{aligned}
{\beta_4}' &= (r^{*\prime} g_1\kappa_1 + r^* g_1'\kappa_1 + r^* g_1\kappa_1')\overrightarrow{T}^* + r^* g_1\kappa_1 \overrightarrow{T}^{*\prime} + (r^{*\prime} h_1\tau_1 + r^* h_1'\tau_1 + r^* h_1\tau_1' - r^{*\prime} g\kappa_1 \\
&\quad -r^* g'\kappa_1 - r^* g\kappa_1')\overrightarrow{N}^* + (r^* h_1\tau_1 - r^* g\kappa_1)\overrightarrow{N}^{*\prime} - (r^{*\prime} g_1\tau_1 + r^* g_1'\tau_1 + r^* g_1\tau_1')\overrightarrow{B}^* \\
&\quad -r^* g_1\tau_1 \overrightarrow{B}^{*\prime} \\
&= (r^{*\prime} g_1\kappa_1 + r^* g_1'\kappa_1 + r^* g_1\kappa_1')\overrightarrow{T}^* + r^* g_1\kappa_1(\kappa_1\overrightarrow{N}^*) + (r^{*\prime} h_1\tau_1 + r^* h_1'\tau_1 + r^* h_1\tau_1' \\
&\quad -r^{*\prime} g\kappa_1 - r^* g'\kappa_1 - r^* g\kappa_1')\overrightarrow{N}^* + (r^* h_1\tau_1 - r^* g\kappa_1)(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) - (r^{*\prime} g_1\tau_1 \\
&\quad +r^* g_1'\tau_1 + r^* g_1\tau_1')\overrightarrow{B}^* - (r^* g_1\tau_1)(-\tau_1\overrightarrow{N}^*)
\end{aligned}
$$

$$
\begin{aligned}
&= (r^{*\prime} g_1\kappa_1 + r^* g_1' \kappa_1 + r^* g_1 \kappa_1' - r^* h_1 \kappa_1 \tau_1 + r^* g \kappa_1^2)\overrightarrow{T}^* + (r^* g_1 \kappa_1^2 + r^{*\prime} h_1 \tau_1 \\
&\quad + r^* h_1' \tau_1 + r^* h_1 \tau_1' - r^{*\prime} g \kappa_1 - r^* g' \kappa_1 - r^* g \kappa_1' + r^* g_1 \tau_1^2)\overrightarrow{N}^* + (r^* h_1 \tau_1^2 - r^* g \kappa_1 \tau_1 \\
&\quad - r^{*\prime} g_1 \tau_1 - r^* g_1' \tau_1 - r^* g_1 \tau_1')\overrightarrow{B}^* \\
&= g_1(r^*\kappa_1)'\overrightarrow{T}^* + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)\overrightarrow{N}^* - g_1(r^*\tau_1)'\overrightarrow{B}^*
\end{aligned}
$$

eşitliği bulunur. Buradan norm alınırsa

$$
\|\beta_4'\| = \left((g_1(r^*\kappa_1)')^2 + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)^2 + (g_1(r^*\tau_1)')^2\right)^{\frac{1}{2}}
$$

olur. Buradan β_4 eğrisinin teğet vektörü

$$
\overrightarrow{T}_{\beta_4} = \frac{g_1(r^*\kappa_1)'\overrightarrow{T}^* + \left(h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1\right)\overrightarrow{N}^* - g_1(r^*\tau_1)'\overrightarrow{B}^*}{\left((g_1(r^*\kappa_1)')^2 + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)^2 + (g_1(r^*\tau_1)')^2\right)^{\frac{1}{2}}}
$$

şeklindedir. β_4' ifadesinin türevi alınırsa

$$
\begin{aligned}
\beta_4'' &= (g_1'(r^*\kappa_1)' + g_1(r^*\kappa_1)'')\overrightarrow{T}^* + g_1(r^*\kappa_1)'\overrightarrow{T}^{*\prime} + \big(h_1'(r^*\tau_1)' + h_1(r^*\tau_1)'' - g'(r^*\kappa_1)' \\
&\quad - g(r^*\kappa_1)'' - (r^*\kappa_1)'\big)\overrightarrow{N}^* + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)\overrightarrow{N}^{*\prime} - (g_1'(r^*\tau_1)' + g_1(r^*\tau_1)'')\overrightarrow{B}^* \\
&\quad - g_1(r^*\tau_1)'\overrightarrow{B}^{*\prime} \\
&= (g_1'(r^*\kappa_1)' + g_1(r^*\kappa_1)'')\overrightarrow{T}^* + (g_1(r^*\kappa_1)')(\kappa_1\overrightarrow{N}^*) + \big(h_1'(r^*\tau_1)' + h_1(r^*\tau_1)'' - g'(r^*\kappa_1)' \\
&\quad - g(r^*\kappa_1)'' - (r^*\kappa_1)'\big)\overrightarrow{N}^* + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) - (g'(r^*\tau_1)' \\
&\quad + g_1(r^*\tau_1)'')\overrightarrow{B}^* - (g_1(r^*\tau_1)')(-\tau_1\overrightarrow{N}^*)
\end{aligned}
$$

$$
\begin{aligned}
&= \big(g_1'(r^*\kappa_1)' + g_1(r^*\kappa_1)'' - \kappa_1(h_1(r^*\tau_1)' - g(r^*\kappa_1)') + r^*\kappa_1^2\big)\overrightarrow{T}^* \\
&\quad +\big(g_1\kappa_1(r^*\kappa_1)' + h_1'(r^*\tau_1)' + h_1(r^*\tau_1)'' - g'(r^*\kappa_1)' - g(r^*\kappa_1)'' - (r^*\kappa_1)' + g_1\tau_1(r^*\tau_1)'\big)\overrightarrow{N}^* \\
&\quad +\big(\tau_1(h_1(r^*\tau_1)' - g(r^*\kappa_1)') - g_1'(r^*\tau_1)' - g_1(r^*\tau_1)'' - r^*\kappa_1\tau_1\big)\overrightarrow{B}^* \\
&= ((h_1\tau_1 - g\kappa_1)(r^*\kappa_1)' + g_1(r^*\kappa_1)'' - \kappa_1(h_1(r^*\tau_1)' - g(r^*\kappa_1)') + r^*\kappa_1^2)\overrightarrow{T}^* + (g_1\kappa_1(r^*\kappa_1)' \\
&\quad -(g_1\tau_1)(r^*\tau_1)' + h_1(r^*\tau_1)'' - (1 + g_1\kappa_1)(r^*\kappa_1)' - g(r^*\kappa_1)'' - (r^*\kappa_1)' + g_1\tau_1(r^*\tau_1)')\overrightarrow{N}^* \\
&\quad +(\tau_1(h_1(r^*\tau_1)' - g(r^*\kappa_1)') - (h_1\tau_1 - g\kappa_1)(r^*\tau_1)' - g_1(r^*\tau_1)'' - r^*\kappa_1\tau_1)\overrightarrow{B}^* \\
&= (g_1(r^*\kappa_1)'' + r^*\kappa_1^2 + h_1r^*(\kappa_1'\tau_1 - \kappa_1\tau_1'))\overrightarrow{T}^* - (g(r^*\kappa_1)'' + 2(r^*\kappa_1)' - h_1(r^*\tau_1)'')\overrightarrow{N}^* \\
&\quad +(gr^*(\kappa_1\tau_1' - \kappa_1'\tau_1) - r^*\kappa_1\tau_1 - g_1(r^*\tau_1)'')\overrightarrow{B}^*
\end{aligned}
$$

olur. Burada

$$
\begin{aligned}
H_1 &= g_1(r^*\kappa_1)'' + r^*\kappa_1^2 + h_1r^*(\kappa_1'\tau_1 - \kappa_1\tau_1') \\
H_2 &= -(g(r^*\kappa_1)'' + 2(r^*\kappa_1)' - h_1(r^*\tau_1)'') \\
H_3 &= gr^*(\kappa_1\tau_1' - \kappa_1'\tau_1) - r^*\kappa_1\tau_1 - g_1(r^*\tau_1)''
\end{aligned}
$$

alınırsa

$$
\beta_4'' = H_1\overrightarrow{T}^* + H_2\overrightarrow{N}^* + H_3\overrightarrow{B}^* \tag{3.1.15}
$$

olur. β_4' ve β_4'' vektörel çarpıldığında

$$\beta_4' \wedge \beta_4'' = \Big((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1)\Big)\overrightarrow{T}^* - (g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)'))\overrightarrow{N}^*$$

$$+\Big((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1)\Big)\overrightarrow{B}^* \qquad (3.1.16)$$

eşitliği bulunur. Bu ifadenin normu alınırsa

$$\|\beta_4' \wedge \beta_4''\| = \bigg(\Big((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1)\Big)^2 + \Big(g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)')\Big)^2$$

$$+\Big((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1)\Big)^2\bigg)^{\frac{1}{2}} \qquad (3.1.17)$$

olur. (2.1.1) eşitliğinde (3.1.16) ve (3.1.17) yerlerine yazıldığında eğrinin binormal vektörü

$$\overrightarrow{B}_{\beta_4} = \frac{\begin{array}{l}\Big((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1)\Big)\overrightarrow{T}^* - \Big(g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)')\Big)\overrightarrow{N}^* \\ +\Big((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1)\Big)\overrightarrow{B}^*\end{array}}{\begin{array}{l}\big(((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1))^2 + (g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)'))^2 \\ +((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1))^2\big)^{\frac{1}{2}}\end{array}}$$

bulunur. (2.1.1) eşitliğinde $\overrightarrow{T}_{\beta_4}$ ve $\overrightarrow{B}_{\beta_4}$ vektörleri yerlerine yazıldığında β_4 eğrisinin asli normal vektörü

$$\overrightarrow{N}_{\beta_4} = \frac{\begin{aligned}&\Big(g_1(r^*\tau_1)'(g_1(H_1(r^*\tau_1)'+H_3(r^*\kappa_1)'))-(h_1(r^*\tau_1)'-g(r^*\kappa_1)'-r^*\kappa_1)((r^*\kappa_1)'\\&(gH_1+g_1H_2)-H_1(h_1(r^*\tau_1)'-r^*\kappa_1))\Big)\overrightarrow{T}^*+\Big(g_1(r^*\kappa_1)'((r^*\kappa_1)'(gH_1+g_1H_2)\\&-H_1(h_1(r^*\tau_1)'-r^*\kappa_1))+g_1(r^*\tau_1)'((r^*\tau_1)'(h_1H_3+g_1H_2)-H_3(g(r^*\kappa_1)'+r^*\kappa_1))\Big)\overrightarrow{N}^*\\&+\Big((h_1(r^*\tau_1)'-g(r^*\kappa_1)'-r^*\kappa_1)((r^*\tau_1)'(h_1H_3+g_1H_2)-H_3(g(r^*\kappa_1)'+r^*\kappa_1))\\&+g_1(r^*\kappa_1)'(g_1(H_1(r^*\tau_1)'+H_3(r^*\kappa_1)'))\Big)\overrightarrow{B}^*\end{aligned}}{\begin{aligned}&\Big(((r^*\tau_1)'(h_1H_3+g_1H_2)-H_3(g(r^*\kappa_1)'+r^*\kappa_1))^2+(g_1(H_1(r^*\tau_1)'+H_3(r^*\kappa_1)'))^2\\&+((r^*\kappa_1)'(gH_1+g_1H_2)-H_1(h_1(r^*\tau_1)'-r^*\kappa_1))^2\Big)^{\frac{1}{2}}.\\&\Big((g_1(r^*\kappa_1')')^2+(h_1(r^*\tau_1)'-g(r^*\kappa_1)'-r^*\kappa_1)^2+(g_1(r^*\tau_1)')^2\Big)^{\frac{1}{2}}\end{aligned}}$$

elde edilir. □

Teorem 3.1.8 β_1 eğrisinin eğriliği κ_{β_1} ve torsiyonu τ_{β_1} ile gösterilsin. Bu eğriliklerin involüt eğrisinin eğrilikleri cinsinden karşılıkları

$$\kappa_{\beta_4} = \frac{\begin{aligned}&\Big(((r^*\tau_1)'(h_1H_3+g_1H_2)-H_3(g(r^*\kappa_1)'+r^*\kappa_1))^2-(g_1(H_1(r^*\tau_1)'+H_3(r^*\kappa_1)'))^2\\&+((r^*\kappa_1)'(gH_1+g_1H_2)-H_1(h_1(r^*\tau_1)'-r^*\kappa_1))^2\Big)^{\frac{1}{2}}\end{aligned}}{\Big((g_1(r^*\kappa_1')')^2+(h_1(r^*\tau_1)'-g(r^*\kappa_1)'-r^*\kappa_1)^2+(g_1(r^*\tau_1)')^2\Big)^{\frac{3}{2}}}$$

$$\tau_{\beta_4} = \frac{\begin{aligned}&g_1(r^*\kappa_1)'\Big(H_2^2\tau_1+H_2H_3'-H_1H_3\kappa_1-H_2'H_3+H_3^2\tau_1\Big)-g_1(r^*\tau_1)'\Big(H_1^2\kappa_1+H_1H_2'\\&-H_1H_3\tau_1-H_1'H_2+H_2^2\kappa_1\Big)+\Big(h_1(r^*\tau_1)'-g(r^*\kappa_1)'-r^*\kappa_1\Big)\Big(H_1'H_3-H_2H_3\kappa_1\\&-H_1H_2\tau_1-H_1H_3'\Big)\end{aligned}}{\begin{aligned}&\Big((r^*\tau_1)'(h_1H_3+g_1H_2)-H_3(g(r^*\kappa_1)'+r^*\kappa_1)\Big)^2+\Big(g_1(H_1(r^*\tau_1)'+H_3(r^*\kappa_1)')\Big)^2\\&+\Big((r^*\kappa_1)'(gH_1+g_1H_2)-H_1(h_1(r^*\tau_1)'-r^*\kappa_1)\Big)^2\end{aligned}}$$

bağıntılarıyla verilir. Burada r^*, H_1, H_2, H_3

$$
\begin{aligned}
H_1 &= g_1(r^*\kappa_1)'' + r^*\kappa_1^2 + h_1 r^*(\kappa_1'\tau_1 - \kappa_1\tau_1') \\
H_2 &= -(g(r^*\kappa_1)'' + 2(r^*\kappa_1)' - h_1(r^*\tau_1)'') \\
H_3 &= gr^*(\kappa_1\tau_1' - \kappa_1'\tau_1) - r^*\kappa_1\tau_1 - g_1(r^*\tau_1)'' \\
r^* &= \frac{1}{\sqrt{\kappa^2+\tau^2}}
\end{aligned}
$$

şeklinde birer katsayıdır.

İspat. (2.1.2) ifadesinde β_4' ve (3.1.17) eşitlikleri yerlerine yazıldığında eğrinin eğriliği

$$
\kappa_{\beta_4} = \frac{\begin{array}{l}\Big(((r^*\tau_1)'(h_1H_3+g_1H_2) - H_3(g(r^*\kappa_1)'+r^*\kappa_1))^2 - (g_1(H_1(r^*\tau_1)'+H_3(r^*\kappa_1)'))^2 \\ +((r^*\kappa_1)'(gH_1+g_1H_2) - H_1(h_1(r^*\tau_1)'-r^*\kappa_1))^2\Big)^{\frac{1}{2}}\end{array}}{\Big((g_1(r^*\kappa_1')')^2 + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1)^2 + (g_1(r^*\tau_1)')^2\Big)^{\frac{3}{2}}}
$$

bulunur. (3.1.15) eşitliğinin türevi alınırsa

$$
\begin{aligned}
\beta_4''' &= H_1\overrightarrow{T}^* + H_1\overrightarrow{T}^{*\prime} + H_2'\overrightarrow{N}^* + H_2\overrightarrow{N}^{*\prime} + H_3'\overrightarrow{B}^* + H_3\overrightarrow{B}^{*\prime} \\
&= H_1\overrightarrow{T}^* + H_1(\kappa_1\overrightarrow{N}^*) + H_2'\overrightarrow{N}^* + H_2(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) + H_3'\overrightarrow{B}^* + H_3(-\tau_1\overrightarrow{N}^*) \\
&= (H_1' - H_2\kappa_1)\overrightarrow{T}^* + (H_1\kappa_1 + H_2' - H_3\tau_1)\overrightarrow{N}^* + (H_2\tau_1 + H_3')\overrightarrow{B}^*
\end{aligned}
$$

olur. β_4', β_4'' ve β_4''' ifadelerinin determinantı

$$
\begin{aligned}
det(\beta_4', \beta_4'', \beta_4'') \;=\;& g_1(r^*\kappa_1)'(H_2^2\tau_1 + H_2H_3' - H_1H_3\kappa_1 - H_2'H_3 + H_3^2\tau_1) - g_1(r^*\tau_1)' \\
& (H_1^2\kappa_1 + H_1H_2' - H_1H_3\tau_1 - H_1'H_2 + H_2^2\kappa_1) + (h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1) \\
& (H_1'H_3 - H_2H_3\kappa_1 - H_1H_2\tau_1 - H_1H_3') \qquad (3.1.18)
\end{aligned}
$$

şeklindedir. (2.1.2) eşitliğinde (3.1.17) ve (3.1.18) ifadeleri yerlerine yazılırsa eğrinin torsiyonu

$$
\tau_{\beta_4} = \frac{\begin{array}{l} g_1(r^*\kappa_1)'\Big(H_2^2\tau_1 + H_2H_3' - H_1H_3\kappa_1 - H_2'H_3 + H_3^2\tau_1\Big) - g_1(r^*\tau_1)'\Big(H_1^2\kappa_1 + H_1H_2' \\ -H_1H_3\tau_1 - H_1'H_2 + H_2^2\kappa_1\Big) + \Big(h_1(r^*\tau_1)' - g(r^*\kappa_1)' - r^*\kappa_1\Big)\Big(H_1'H_3 - H_2H_3\kappa_1 \\ -H_1H_2\tau_1 - H_1H_3'\Big) \end{array}}{\begin{array}{l} \Big((r^*\tau_1)'(h_1H_3 + g_1H_2) - H_3(g(r^*\kappa_1)' + r^*\kappa_1)\Big)^2 + \Big(g_1(H_1(r^*\tau_1)' + H_3(r^*\kappa_1)')\Big)^2 \\ +\Big((r^*\kappa_1)'(gH_1 + g_1H_2) - H_1(h_1(r^*\tau_1)' - r^*\kappa_1)\Big)^2 \end{array}}
$$

elde edilir. □

Sonuç 3.1.7 β_4 eğrisinin $\overrightarrow{T}_{\beta_4}, \overrightarrow{N}_{\beta_4}, \overrightarrow{B}_{\beta_4}$ Frenet vektörlerinin α eğrisinin Frenet vektörleri cinsinden eşitleri

$$
\overrightarrow{T}_{\beta_4} = \frac{\begin{array}{l} \Big(\frac{g\kappa}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})' + \frac{r^*}{\lambda} - h(\frac{r^*\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)})'\Big)\overrightarrow{T} + \big(h\tau - \kappa(f+\lambda)\big)'. \\ (\frac{r^*\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)})'\overrightarrow{N} + \big((f+\lambda)(\frac{r^*\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)})' - \frac{g\tau}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})' - \frac{r^*\tau}{\lambda\kappa}\big)\overrightarrow{B} \end{array}}{\begin{array}{l} \Big(\big(h\tau - \kappa(f+\lambda)\big)^2\big((\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\big)^2 + \big((h\kappa + \tau(f+\lambda))(\frac{(\frac{\tau}{\kappa})'\kappa^2}{\lambda\kappa(\kappa^2+\tau^2)})' \\ -g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})' - \frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)^2 + \big((h\tau - \kappa(f+\lambda))(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\big)^2\Big)^{\frac{1}{2}} \end{array}}
$$

$$\overrightarrow{N}_{\beta_4} = \frac{K_1\overrightarrow{T}+K_2\overrightarrow{N}+K_3\overrightarrow{B}}{\Big(M_1^2+M_2^2+M_3^2\Big)^{\frac{1}{2}}\Big(\big(h\tau-\kappa(f+\lambda)\big)^2\big((\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\big)^2 + \big((h\kappa+\tau(f+\lambda))(\frac{(\frac{\tau}{\kappa})'\kappa^2}{\lambda\kappa(\kappa^2+\tau^2)})'-g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)^2 + \big((h\tau-\kappa(f+\lambda))(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\big)^2\Big)^{\frac{1}{2}}}$$

$$\overrightarrow{B}_{\beta_4} = \frac{\overrightarrow{T}+\overrightarrow{N}+\overrightarrow{B}}{\Big(M_1^2+M_2^2+M_3^2\Big)^{\frac{1}{2}}}$$

bağıntılarıyla verilir. Burada H_1, H_2, H_3, K_1, K_2, K_3, M_1, M_2, M_3, P_1, P_2 ve P_3

$$H_1 = \frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)})''-\frac{(\frac{\tau}{\kappa})'}{\lambda^2\sqrt{\kappa^2+\tau^2}}+(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}$$

$$+\frac{r^*(\kappa^2+\tau^2)}{\lambda^2\kappa^2}+\frac{r^*(h\kappa+\tau(f+\lambda))}{\kappa^2+\tau^2}$$

$$H_2 = -g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''-(\frac{2r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})''$$

$$H_3 = gr^*\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'-(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big)-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}$$

$$-\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})''$$

$$r^* = \frac{\lambda(\kappa^2+\tau^2)}{\sqrt{(\kappa^2+\tau^2)^3+((\frac{\tau}{\kappa})')^2\kappa^2}}$$

şeklinde birer kaysayıdır.

$$
\begin{aligned}
K_1 &= \frac{\kappa^2(f+\lambda)-h\kappa\tau}{\kappa^2+\tau^2}\Big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'\Big((\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\big(gH_1+H_2\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}})\\
&\quad -H_1\big(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\Big)+\frac{\kappa^2(f+\lambda)-h\kappa\tau}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'.\\
&\quad \Big((\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_3+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2\big)-H_3\big(g(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\\
&\quad -\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\Big)+\Big(\frac{h\kappa\tau+\tau^2(f+\lambda)}{\kappa^2+\tau^2}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'-\frac{g\tau}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'-\frac{r^*\tau}{\lambda\kappa}\Big).\\
&\quad \Big((\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_3+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2\big)-H_3\big(g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\\
&\quad -\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\Big)-\frac{h\tau^2-\kappa\tau(f+\lambda)}{\kappa^2+\tau^2}(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\kappa^2+\tau^2})'\Big(\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(H_1(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\\
&\quad +H_3(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})')\Big)
\end{aligned}
$$

$$
\begin{aligned}
K_2 &= \Big(H_1(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+H_3(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\Big)\Big((\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'(\frac{(h\tau-\kappa(f+\lambda))^2}{\kappa^2+\tau^2}\\
&\quad +\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}})-g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)-\Big(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\\
&\quad -g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)\Big((\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\big(gH_1+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
K_3 &= \frac{h\tau^2-\kappa\tau(f+\lambda)}{\kappa^2+\tau^2}\Big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'\Big(\Big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big)'\big(gH_1+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2\big)\\
&-H_1\big(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)\Big)+\frac{h\tau^2-\kappa\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'.\\
&\Big(\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\big(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_3+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2\big)-H_3\big(g\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\\
&-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)\Big)+\Big(\frac{h\kappa^2+\kappa\tau(f+\lambda)}{\kappa^2+\tau^2}\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'-\frac{g\kappa}{\sqrt{\kappa^2+\tau^2}}\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'-\frac{r^*}{\lambda}\Big).\\
&\Big(\big(\frac{(r^*\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\big(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_3+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2\big)-H_3\big(g\big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\\
&-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)\Big)-\frac{h\kappa\tau-\kappa^2(f+\lambda)}{\kappa^2+\tau^2}\big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\kappa^2+\tau^2}\big)'\Big(\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(H_1\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\\
&+H_3\big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\big)\Big)
\end{aligned}
$$

$$
M_1 = \frac{\kappa(f+\lambda)-h\tau}{\sqrt{\kappa^2+\tau^2}}\Big(H_1\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'+H_3\big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\Big)
$$

$$
\begin{aligned}
M_2 &= \big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\Big(\frac{h(H_3\kappa+H_2\tau)+(f+\lambda)(H_3\tau-H_2\kappa)}{\sqrt{\kappa^2+\tau^2}}\Big)\\
&-H_3\big(g\big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= \big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\big(gH_1+\frac{h\tau-\kappa(f+\lambda)}{\kappa^2+\tau^2}H_2\big)'-H_1\Big(\frac{h\kappa+\tau(f+\lambda)}{\kappa^2+\tau^2}.\\
&\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= \frac{h\kappa-\kappa^2(f+\lambda)}{\kappa^2+\tau^2}\Big(H_1(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+H_3(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})\Big)+\frac{\tau}{\sqrt{\kappa^2+\tau^2}}(r^*\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'.\\
&\quad \big(gH_1+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2\big)-H_1\tau\big(\frac{(h\kappa+\tau(f+\lambda))}{\kappa^2+\tau^2}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'-\frac{r^*}{\lambda\kappa}\big)
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= (\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\Big(\frac{h(H_2\tau+H_3\kappa)+(f+\lambda)(H_3\tau-H_2\kappa)}{\sqrt{\kappa^2+\tau^2}}\Big)-H_3\Big(g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\\
&\quad -\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)
\end{aligned}
$$

$$
\begin{aligned}
P_3 &= \frac{\kappa\tau(f+\lambda)-h\tau^2}{\kappa^2+\tau^2}\Big(H_1(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+H_3(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\Big)+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'.\\
&\quad \big(gH_1+\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}H_2\big)-H_1\kappa\big(\frac{h\kappa+\tau(f+\lambda)}{\kappa^2+\tau^2}-\frac{r^*}{\lambda}\big)
\end{aligned}
$$

Sonuç 3.1.8 β_4 eğrisinin κ_{β_4} ve τ_{β_4} eğriliklerinin α esas eğrinin eğrilikleri cinsinden yazılışı

$$
\begin{aligned}
\kappa_{\beta_4} &= \frac{\Big(M_1^2+M_2^2+M_3^2\Big)^{\frac{1}{2}}}{\begin{array}{l}\Big(\big(h\tau-\kappa(f+\lambda)\big)^2\big((\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\big)^2+\big((h\kappa+\tau(f+\lambda))(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\\
-g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'-\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)^2+\big((h\tau-\kappa(f+\lambda))(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'\big)^2\Big)^{\frac{3}{2}}\end{array}}
\end{aligned}
$$

$$
\tau_{\beta_4} = \frac{\begin{array}{l}\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\Big(\frac{H_2^2(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}+H_2H_3'-H_1H_3\kappa_1-H_2'H_3\\
+\frac{H_3^2(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big)-\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'+\Big(H_1^2\kappa_1H_1H_2'-\frac{H_1H_3(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\\
-H_1'H_2+\frac{H_2^2\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)+\Big(\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})'-g\big(r^*\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\\
-r^*\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)\Big(H_1'H_3-\frac{H_2H_3\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}-\frac{H_1H_2(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}-H_1H_3'\Big)\end{array}}{M_1^2+M_2^2+M_3^2}
$$

bağıntılarıyla verilir. Burada H_1, H_2, H_3, M_1, M_2 ve M_3

$$
\begin{aligned}
H_1 &= \frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*\kappa(\frac{\tau}{\kappa})'}{\lambda(\kappa^2+\tau^2)})''-\frac{(\frac{\tau}{\kappa})'}{\lambda^2\sqrt{\kappa^2+\tau^2}}+(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\\
&\quad+\frac{r^*(\kappa^2+\tau^2)}{\lambda^2\kappa^2}+\frac{r^*(h\kappa+\tau(f+\lambda))}{\kappa^2+\tau^2}
\end{aligned}
$$

$$
H_2 = -g(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''-2(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'+\frac{h\kappa+\tau(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)})''
$$

$$
\begin{aligned}
H_3 &= gr^*\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big(\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)' - \big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\Big) - \frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)} \\
&\quad -\frac{h\tau-\kappa(f+\lambda)}{\sqrt{\kappa^2+\tau^2}}\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)''
\end{aligned}
$$

$$
r^* = \frac{\lambda(\kappa^2+\tau^2)}{\sqrt{(\kappa^2+\tau^2)^3+((\frac{\tau}{\kappa})')^2\kappa^2}}
$$

$$
M_1 = \frac{\kappa(f+\lambda)-h\tau}{\sqrt{\kappa^2+\tau^2}}\Big(H_1\big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)' + H_3\big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\Big)
$$

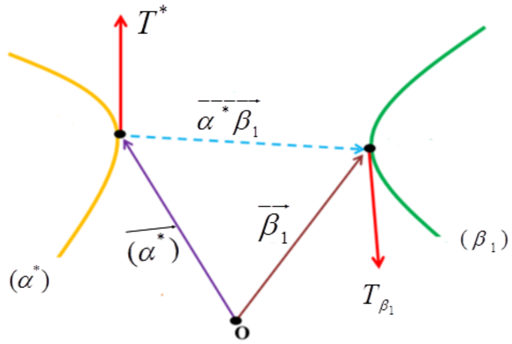
$$
\begin{aligned}
M_2 &= \big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)'\Big(\frac{h(H_3\kappa+H_2\tau)+(f+\lambda)(H_3\tau-H_2\kappa)}{\sqrt{\kappa^2+\tau^2}}\Big) \\
&\quad -H_3\big(g\big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)' - \frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= \big(\frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\big)'\big(gH_1+\frac{h\tau-\kappa(f+\lambda)}{\kappa^2+\tau^2}H_2\big)' - H_1\Big(\frac{h\kappa+\tau(f+\lambda)}{\kappa^2+\tau^2}. \\
&\quad \big(\frac{r^*(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\big)' - \frac{r^*\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)
\end{aligned}
$$

şeklinde birer katsayıdır.

3.2 İnvolüt Eğrisinin Vektörel Moment Eğrileriyle Sabit Genişlikli Eğri Çifti Olma Durumları

Teorem 3.2.1 α eğrisinin involütü α^*, α^* eğrisinin $\overrightarrow{T}^*$ teğet vektörünün vektörel moment vektörü $\overrightarrow{V}_1 = \alpha^* \wedge \overrightarrow{T}^*$ olsun. $\overrightarrow{V}_1$ vektörünün çizdiği β_1 eğrisi ile α^* eğrisi sabit genişlikli eğri çifti oluşturur (Şekil 3.5).



Şekil 3.5: α^* İnvolüt Eğrisiyle T^* Vektörünün Vektörel Momentinin Çizdiği β_1 Eğrisi

İspat. α^* eğrisi Frenet aparatlarına bağlı olarak

$$\alpha^* = f_1\overrightarrow{T}^* + g_1\overrightarrow{N}^* + h_1\overrightarrow{B}^* \tag{3.2.1}$$

şeklinde yazılır. $\overrightarrow{V_1}$ vektörel moment vektörü

$$\begin{aligned} \overrightarrow{V_1} &= \alpha^* \wedge \overrightarrow{T}^* \\ &= (f_1\overrightarrow{T}^* + g_1\overrightarrow{N}^* + h_1\overrightarrow{B}^*) \wedge \overrightarrow{T}^* \\ &= h_1\overrightarrow{N}^* - g_1\overrightarrow{B}^* \end{aligned}$$

olur. Bu vektörün çizdiği eğri β_1 ile gösterildiğinde

$$\beta_1 = h_1\overrightarrow{N}^* - g_1\overrightarrow{B}^* \tag{3.2.2}$$

olur. $\overrightarrow{\alpha^*\beta_1}$ (Şekil 3.5)'den

$$\overrightarrow{\alpha^*\beta_1} = m_1\overrightarrow{T}^* + m_2\overrightarrow{N}^* + m_3\overrightarrow{B}^*$$

eşitliği yazılır. Burada (3.2.1) ve (3.2.2) ifadeleri dikkate alınırsa

$$\begin{aligned}\beta_1 - \alpha^* &= m_1\overrightarrow{T}^* + m_2\overrightarrow{N}^* + m_3\overrightarrow{B}^* \\ h_1\overrightarrow{N}^* - g_1\overrightarrow{B}^* - f_1\overrightarrow{T}^* - g_1\overrightarrow{N}^* - h_1\overrightarrow{B}^* &= m_1\overrightarrow{T}^* + m_2\overrightarrow{N}^* + m_3\overrightarrow{B}^*\end{aligned}$$

olur. Burada gerekli düzenlemeler yapıldığında

$$-f_1\overrightarrow{T}^* + (h_1 - g_1)\overrightarrow{N}^* - (g_1 + h_1)\overrightarrow{B}^* = m_1\overrightarrow{T}^* + m_2\overrightarrow{N}^* + m_3\overrightarrow{B}^*$$

bulunr. Burada m_1,m_2 ve m_3 katsayıları sırasıyla

$$m_1 = -f_1 \quad m_2 = h_1 - g_1 \quad m_3 = -(g_1 + h_1)$$

şeklindedir. Bu katsayılar β_1 eğrisinde yerine yazıldığında

$$\beta_1 = \alpha^* - f_1\overrightarrow{T}^* + (h_1 - g_1)\overrightarrow{N}^* - (g_1 + h_1)\overrightarrow{B}^*$$

olur. α^* eğrisi düzlemsel olduğundan $\tau_1 = 0$ olur. α^* ve β_1 eğrilerinin aynı düzlemde olması için $f_1 = 0$ olmalıdır. Bu katsayılar β_1 eğrisinde yerine yazıldığında

$$\beta_1 = \alpha^* + (h_1 - g_1)\overrightarrow{N}^* - (g_1 + h_1)\overrightarrow{B}^*$$

olur. $\tau_1 = 0$ ve $f_1 = 0$ olduğundan (3.1.3) ifadesinde yerine yazıldığında

$$g_1' = 0 \;\Rightarrow\; g_1 = sabit \quad h_1' = 0 \;\Rightarrow\; h_1 = sabit$$

şeklindedir. (Şekil 3.5)'den α^* ile β_1 arasındaki uzaklık

$$\begin{aligned} d(\alpha^*, \beta_1) = \| \overrightarrow{\alpha^*\beta_1} \|^2 &= (h_1 - g_1)^2 + (h_1 + g_1)^2 \\ &= 2(g_1^2 + h_1^2) \end{aligned}$$

bulunur . Burada uzaklığın sabit olduğu görülür. β_1 eğrisinin s parametresine göre türevi alınırsa

$$\begin{aligned} \frac{d\beta_1}{ds^*}.\frac{ds^*}{ds} &= (\alpha^*)' + (h_1 - g_1)'\overrightarrow{N}^* + (h_1 - g_1)(\overrightarrow{N}^*)' - (g_1 + h_1)'\overrightarrow{B}^* - (g_1 + h_1)(\overrightarrow{B}^*)' \\ \overrightarrow{T}_{\beta_1}.\frac{ds^*}{ds} &= \overrightarrow{T}^* + (h_1 - g_1)'\overrightarrow{N}^* + (h_1 - g_1)(-\kappa_1\overrightarrow{T}^* + \tau_1\overrightarrow{B}^*) \\ &\quad -(g_1 + h_1)'\overrightarrow{B}^* - (g_1 + h_1)(-\tau_1\overrightarrow{N}^*) \\ &= (1 - \kappa_1(h_1 - g_1))\overrightarrow{T}^* + ((h_1 - g_1)' + (h_1 + g_1)\tau_1)\overrightarrow{N}^* + ((h_1 - g_1)\tau_1 - (h_1 + g_1)')\overrightarrow{B}^* \end{aligned}$$

şeklinde olur. $g_1 = sabit, h_1 = sabit$ ve $\tau_1 = 0$ olma şartı dikkate alınırsa

$$\begin{aligned} \overrightarrow{T}_{\beta_1}.\frac{ds^*}{ds} &= (1 - \kappa_1(h_1 - g_1))\overrightarrow{T}^* \\ &= (\underbrace{1 + g_1\kappa_1}_{f_1'} - h_1\kappa_1)\overrightarrow{T}^* \end{aligned}$$

eşitliği bulunur. Burada $f_1 = 0$ olduğundan

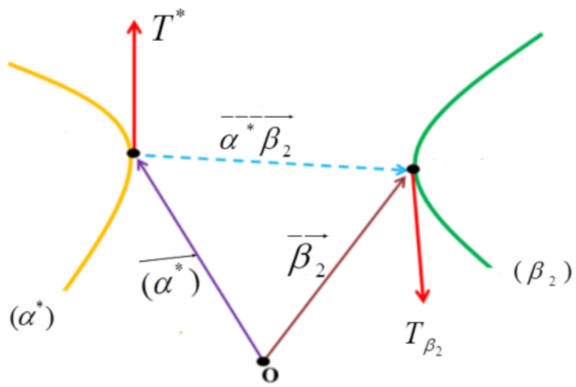
$$T_{\beta_1}.\frac{ds^*}{ds} = -h_1\kappa_1 T^*$$

olur. $\{\alpha^*, \beta_1\}$ sabit genişlikli eğri olduğundan $\overrightarrow{T}^* = -\overrightarrow{T}_{\beta_1}$ olur. Burada

$$h_1\kappa_1 = 1 \quad veya \quad h_1 = \frac{1}{\kappa_1} > 0$$

eşitlikleri elde edilir. Bu durumda teğet vektörler paralel ve zıt yönlü bulunur. α^* ve β_1 eğrilerinin aralarındaki uzaklık sabit ve teğet vektörleri paralel-zıt yönlü olduğundan bu iki eğri sabit genişlikli eğri çifti oluşturur. □

Teorem 3.2.2 α eğrisinin involütü α^*, α^* eğrisinin $\overrightarrow{N}^*$ asli normal vektörünün vektörel moment vektörü $\overrightarrow{V}_2 = \alpha^* \wedge \overrightarrow{N}^*$ olsun. $\overrightarrow{V}_2$ vektörünün çizdiği β_2 eğrisi ile α^* eğrisi sabit genişlikli eğri çifti oluşturmaz (Şekil 3.6).



Şekil 3.6: α^* İnvolüt Eğrisiyle N^* Vektörünün Vektörel Momentinin Çizdiği β_2 Eğrisi

İspat. α^* eğrisi Frenet aparatlarına bağlı olarak

$$\alpha^* = f_1 \overrightarrow{T}^* + g_1 \overrightarrow{N}^* + h_1 \overrightarrow{B}^* \tag{3.2.3}$$

şeklinde yazılır. $\overrightarrow{V_2}$ vektörel moment vektörü

$$\begin{aligned} \overrightarrow{V_2} &= \alpha^* \wedge \overrightarrow{N}^* \\ &= (f_1 \overrightarrow{T}^* + g_1 \overrightarrow{N}^* + h_1 \overrightarrow{B}^*) \wedge \overrightarrow{N}^* \\ &= -h_1 \overrightarrow{T}^* + f_1 \overrightarrow{B}^* \end{aligned}$$

bulunur. Bu vektörün çizdiği eğri β_2 ile gösterildiğinde

$$\beta_2 = -h_1\overrightarrow{T}^* + f_1\overrightarrow{B}^* \tag{3.2.4}$$

olur. $\overrightarrow{\alpha^*\beta_2}$ (Şekil 3.6)'den

$$\overrightarrow{\alpha^*\beta_2} = n_1\overrightarrow{T}^* + n_2\overrightarrow{N}^* + n_3\overrightarrow{B}^*$$

ifadesi yazılır. Burada (3.2.3) ve (3.2.4) ifadeleri dikkate alınırsa

$$\begin{aligned} \beta_2 - \alpha^* &= n_1\overrightarrow{T}^* + n_2\overrightarrow{N}^* + n_3\overrightarrow{B}^* \\ -h_1\overrightarrow{T}^* + f_1\overrightarrow{B}^* - f_1\overrightarrow{T}^* - g_1\overrightarrow{N}^* - h_1\overrightarrow{B}^* &= n_1\overrightarrow{T}^* + n_2\overrightarrow{N}^* + n_3\overrightarrow{B}^* \end{aligned}$$

olur. Burada gerekli düzenlemeler yapıldığında

$$-(f_1 + h_1)\overrightarrow{T}^* - g_1\overrightarrow{N}^* + (f_1 - h_1)\overrightarrow{B}^* = n_1\overrightarrow{T}^* + n_2\overrightarrow{N}^* + n_3\overrightarrow{B}^*$$

bulunur. Bu eşitlikten n_1,n_2 ve n_3 katsayıları sırasıyla

$$n_1 = -(f_1 + h_1) \quad n_2 = -g_1 \quad n_3 = f_1 - h_1$$

şeklindedir. Bu eşitlikler β_2 eğrisinde yerine yazıldığında

$$\beta_2 = \alpha^* - (f_1 + h_1)\overrightarrow{T}^* - g_1\overrightarrow{N}^* + (f_1 - h_1)\overrightarrow{B}^*$$

olur. α^* eğrisi düzlemsel olduğundan $\tau_1 = 0$ olur. α^* ve β_2 eğrilerinin aynı düzlemde olması için $g_1 = 0$ olmalıdır. Bu katsayılar β_2 eğrisinde yerine yazıldığında

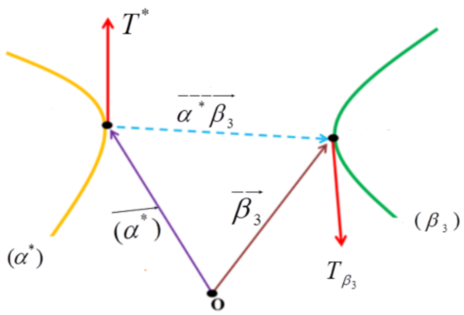
$$\beta_2 = \alpha^* - (f_1 + h_1)\overrightarrow{T}^* + (f_1 - h_1)\overrightarrow{B}^*$$

bulunur. $\tau_1 = 0$ ve $g_1 = 0$ olduğundan (3.1.3) ifadesinin yeni hali

$$h_1' = 0 \Rightarrow \quad h_1 = sabit \quad f_1' = 1 \ \Rightarrow \ \underbrace{f_1 = s} \quad -f_1\kappa_1 = 0 \Rightarrow \underbrace{f_1 = 0}$$

olur. Bu bir çelişkidir. Dolayısıyla bu iki eğri sabit genişlikli eğri değildir. $\square$

Teorem 3.2.3 α eğrisinin involütü α^*, α^* eğrisinin $\overrightarrow{B}^*$ binormal vektörünün vektörel moment vektörü $\overrightarrow{V_3} = \alpha^* \wedge \overrightarrow{B}^*$ olsun. $\overrightarrow{V_3}$ vektörünün çizdiği β_3 eğrisi ile α^* eğrisi sabit genişlikli eğri çifti oluşturmaz (Şekil 3.7).



Şekil 3.7: α^* İnvolüt Eğrisiyle B^* Vektörünün Vektörel Momentinin Çizdiği β_3 Eğrisi

İspat. α^* eğrisi Frenet aparatlarına bağlı olarak

$$\alpha^* = f_1\overrightarrow{T}^* + g_1\overrightarrow{N}^* + h_1\overrightarrow{B}^* \tag{3.2.5}$$

şeklinde yazılır. $\overrightarrow{V_3}$ vektörel moment vektörü

$$\begin{aligned} \overrightarrow{V_3} &= \alpha^* \wedge \overrightarrow{B}^* \\ &= (f_1\overrightarrow{T}^* + g_1\overrightarrow{N}^* + h_1\overrightarrow{B}^*) \wedge \overrightarrow{B}^* \\ &= g_1\overrightarrow{T}^* - f_1\overrightarrow{N}^* \end{aligned}$$

bulunur. Bu vektörün çizdiği eğri β_3 ile gösterildiğinde

$$\beta_3 = g_1\overrightarrow{T}^* - f_1\overrightarrow{N}^* \tag{3.2.6}$$

olur. $\overrightarrow{\alpha^*\beta_3}$ (Şekil 3.7)'den

$$\overrightarrow{\alpha^*\beta_3} = p_1\overrightarrow{T}^* + p_2\overrightarrow{N}^* + p_3\overrightarrow{B}^*$$

eşitliği yazılır. Burada (3.2.5) ve (3.2.6) bağıntıları dikkate alınırsa

$$\begin{aligned} \beta_3 - \alpha^* &= p_1\overrightarrow{T}^* + p_2\overrightarrow{N}^* + p_3\overrightarrow{B}^* \\ g_1\overrightarrow{T}^* - f_1\overrightarrow{N}^* - f_1\overrightarrow{T}^* - g_1\overrightarrow{N}^* - h_1\overrightarrow{B}^* &= p_1\overrightarrow{T}^* + p_2\overrightarrow{N}^* + p_3\overrightarrow{B}^* \end{aligned}$$

olur. Burada gerekli düzenlemeler yapıldığında

$$(g_1 - f_1)\overrightarrow{T}^* - (f_1 + g_1)\overrightarrow{N}^* - h_1\overrightarrow{B}^* = p_1\overrightarrow{T}^* + p_2\overrightarrow{N}^* + p_3\overrightarrow{B}^*$$

bulunur. Bu eşitlikten p_1,p_2 ve p_3 katsayıları sırasıyla

$$p_1 = g_1 - f_1, \quad p_2 = -(f_1 + g_1), \quad p_3 = -h_1$$

şeklinde olur. Bu eşitlikler β_3 eğrisinde yerine yazıldığında

$$\beta_3 = \alpha^* + (g_1 - f_1)\overrightarrow{T}^* - (f_1 + g_1)\overrightarrow{N}^* - h_1\overrightarrow{B}^*$$

eşitliği bulunur. α^* eğrisi düzlemsel olduğundan $\tau_1 = 0$ olur. α^* ve β_3 eğrilerinin aynı düzlemde olması için $h_1 = 0$ olmalıdır. Bu katsayılar β_3 eğrisinde yerine yazılırsa

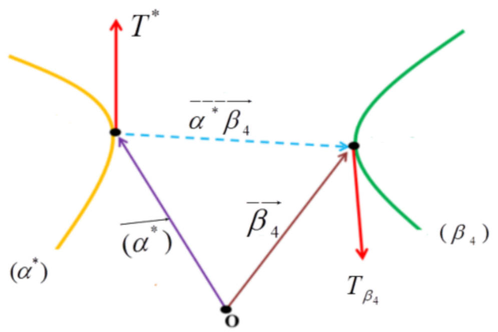
$$\beta_3 = \alpha^* + (g_1 - f_1)\overrightarrow{T}^* - (f_1 + g_1)\overrightarrow{N}^*$$

bağıntısı olur. Buradan α^* ile β_3 arasındaki uzaklık (Şekil 3.7)' den

$$d(\alpha^*, \beta_3) = \| \overrightarrow{\alpha^*\beta_3} \|^2 = (g_1 - f_1)^2 - (f_1 + g_1)^2 = 2(f_1^2 + g_1^2)$$

şeklinde olur. Burada uzaklığın sabit olmadığı görülür. α^* ve β_3 sabit genişlikli eğri kategorisinde değildir. □

Teorem 3.2.4 α eğrisinin involütü α^*, α^* eğrisinin $\overrightarrow{C}^*$ birim Darboux vektörünün vektörel moment vektörü $\overrightarrow{V_4} = \alpha^* \wedge \overrightarrow{C}^*$ olsun. $\overrightarrow{V_4}$ vektörünün çizdiği β_4 eğrisi ile α^* eğrisi sabit genişlikli eğri çifti oluşturmaz (Şekil 3.5).



Şekil 3.8: α^* İnvolüt Eğrisiyle C^* Vektörünün Vektörel Momentinin Çizdiği β_4 Eğrisi

İspat. α^* eğrisi Frenet aparatlarına bağlı olarak

$$\alpha^* = f_1\overrightarrow{T}^* + g_1\overrightarrow{N}^* + h_1\overrightarrow{B}^* \tag{3.2.7}$$

şeklinde yazılır. $\overrightarrow{V_4}$ vektörel moment vektörü $r^* = \frac{1}{\sqrt{\kappa_1^2+\tau_1^2}}$ için

$$\begin{aligned}
\overrightarrow{V_4} &= \alpha^* \wedge \overrightarrow{C}^* \\
&= (f_1\overrightarrow{T}^* + g_1\overrightarrow{N}^* + h_1\overrightarrow{B}^*) \wedge \overrightarrow{C}^* \\
&= r^*g_1\kappa_1\overrightarrow{T}^* + (r^*h_1\tau_1 - r^*f_1\kappa_1)\overrightarrow{N}^* - r^*g_1\tau_1\overrightarrow{B}^*
\end{aligned}$$

şeklinde olur. V_4 vektörün çizdiği eğri β_4 ile gösterildiğinde bu eğri

$$\beta_4 = r^* g_1 \kappa_1 \overrightarrow{T}^* + (r^* h_1 \tau_1 - r^* f_1 \kappa_1)\overrightarrow{N}^* - r^* g_1 \tau_1 \overrightarrow{B}^* \qquad (3.2.8)$$

şeklindedir. $\overrightarrow{\alpha^* \beta_4}$ (Şekil 3.8)'den

$$\overrightarrow{\alpha^* \beta_4} = z_1 \overrightarrow{T}^* + z_2 \overrightarrow{N}^* + z_3 \overrightarrow{B}^*$$

eşitliği yazılır. Burada

$$\beta_4 - \alpha^* = z_1 \overrightarrow{T}^* + z_2 \overrightarrow{N}^* + z_3 \overrightarrow{B}^*$$

bağıntısı mevcuttur. Bu eşitlikte (3.2.7) ve (3.2.8) ifadeleri yazıldığında

$$r^* g_1 \kappa_1 \overrightarrow{T}^* + (r^* h_1 \tau_1 - r^* f_1 \kappa_1)\overrightarrow{N}^* - r^* g_1 \tau_1 \overrightarrow{B}^* - f_1 \overrightarrow{T}^* - g_1 \overrightarrow{N}^* - h_1 \overrightarrow{B}^* = z_1 \overrightarrow{T}^* + z_2 \overrightarrow{N}^* + z_3 \overrightarrow{B}^*$$

eşitliği bulunur. Bu eşitlikten z_1,z_2 ve z_3 katsayıları sırasıyla

$$z_1 = r^* g_1 \kappa_1 - f_1 \qquad z_2 = r^*(h_1 \tau_1 - f_1 \kappa_1) - g_1 \qquad z_3 = -r^* g_1 \tau_1 - h_1$$

şeklinde olur. Bu eşitlikler β_4 eğrisinde yerine yazıldığında bu eğri

$$\beta_4 = \alpha^* + (r^* g_1 \kappa_1 - f_1)\overrightarrow{T}^* + r^*(h_1 \tau_1 - f_1 \kappa_1 - g_1)\overrightarrow{N}^* + (r^* g_1 \tau_1 + h_1)\overrightarrow{B}^*$$

biçimindedir. α^* eğrisi düzlemsel olduğundan $\tau_1 = 0$ olur. Buna göre β_4 eğrisi düzenlendiğinde

$$\beta_4 = \alpha^* + (g_1 - f_1)\overrightarrow{T}^* - (f_1 + g_1)\overrightarrow{N}^* + h_1 \overrightarrow{B}^*$$

olur. (Şekil 3.8) den α^* ile β_4 arasındaki uzaklık

$$d(\alpha^*, \beta_4) = \| \overrightarrow{\alpha^* \beta_4} \|^2 = 2(f_1^2 + g_1^2) + h_1^2$$

şeklindedir. Buradan uzaklığın sabit olmadığı görülür. α^* ve β_4 sabit genişlikli eğri kategorisinde değildir. □

4. SONUÇ ve ÖNERİLER

Bu tezde elde edilen sonuçlar bulgular bölümünde şekillerle açıklanmıştır. Burada ilk olarak, involüt eğrisinin Frenet vektörileri ve Darboux vektörünün vektörel moment vektörleri tanımlandı. Daha sonra moment vektörlerinin çizdiği eğrilerin Frenet vektörleri, eğrilik ve burulması hesaplandı. Bu eğrilerin esas eğrinin Frenet aparatları cinsinden ifadesi verildi. Son olarak bu eğrilerin involüt eğrisiyle sabit genişlikli eğri çiftine dahil olup olmadıkları araştırıldı.

Bu çalışmanın benzeri evolüt eğrileri, Successor eğrileri, Betrand ve Mannheim eğri çiftleri için yeniden yapılabilir. Bu eğriler üzerinde çatılar değiştirilerek vektörel moment eğrileri tekrar tanımlanabilir. Lorentz ve Galileo uzayında da moment eğrileri tanımlanabilir ve gerekli sonuçlara oluşabilir.

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